Fundamentals of Fluid Mechanics

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1 Fundamentals of hydraulics (hydromechanics)

Mechanics generally focuses on the behaviour of bodies under the influence of forces. Bodies consist of materials with defined characteristics. These materials can occur in three states of aggregation: solid, liquid and gaseous. Hydromechanics is the subsection of mechanics, which is concerned particularly with the behaviour of liquids and gases.

1.1 Characteristics of fluids

1.1.1 What are fluids?

Liquids and gases are summarized under the term fluids. Their behaviour under shear stress is very similar. However, under normal (compressive -) stress they behave partially very differently. Liquids hardly change their volume under pressure. In practice liquids are considered as slightly compressible or incompressible. In contrast, gases are strongly compressible. They already show large variations in volume with small differences of pressure. In some practical applications, however, the differences of pressure are so small that in these cases one can consider the gas flow as incompressible (e.g. currents on buildings, motor vehicles or aeroplanes in the subsonic region).

1.1.2 Difference: Solid and fluid state

Solids and fluids can be differentiated from each other by their deformation characteristics due to the effect of shear stress.

1.1.2.1 Effect of shear stress on solids

Immediately after applying shear stress, a solid body deforms by a certain amount $\alpha$ (see Figures 1.1 and 1.2). At the end of the deformation a static equilibrium occurs. If the influencing shear stress is not changed, the body remains in the given deformed state. The shear stress $\tau_{xy}$ is proportional to the deformation size $\alpha$. This relationship can be defined, using a proportionality factor depending on the material, (equation (1.1)) and is well-known as Hook's law. The proportionality factor G is the shear modulus - a material constant.
Hook's law: \[ \tau_{xy} = G \cdot \alpha \]  

(1.1)

### 1.1.2.2 Effect of shear stress on fluids

A fluid (exception: Bingham fluid) deforms continuously under the influence of shear stress. No static equilibrium occurs. As long as shear stress influences, deformation takes place. The shear stress \( \tau_{xy} \) is proportional to the deformation rate \( \frac{d\alpha}{dt} \), as was determined empirically by experimentation. This relationship can be defined, using a fluid-dependent proportionality factor, as equation (1.2). The proportionality factor \( \mu \) is the (dynamic) viscosity or tenacity - a material constant.

\[ \tau_{xy} = \mu \cdot \frac{d\alpha}{dt} \]  

(1.2)

The equation (1.2) can be rearranged by the following transformations into Newton's Hypothesis for shear stress:
Fig. 1.5: Translation and deformation of a fluid element

Thereby: \( dy = \) height of the fluid element

\( ds = \frac{dv}{dt} \cdot \text{Deformation size} \)

Deformation: \( \tan(d\alpha) = \frac{ds}{dy} \) (1.3)

For small angles \( \alpha \):

\[ d\alpha = \frac{ds}{dy} \] (1.4)

\[ d\alpha = \frac{dv}{dt} \frac{dy}{dy} \] (1.5)

\[ \frac{d\alpha}{dt} = \frac{dv}{dy} \] (1.6)

If the deformation rate \( d\alpha/dt \) is replaced by the so-called speed gradients \( dv/dy \) in equation (1.2) then **Newton's hypothesis for shear stress (Newton's friction law)** is obtained.

**Newton's hypothesis:** \( \tau_{xy} = \mu \cdot \frac{dv}{dy} \) (1.7)

### 1.1.3 Relevant characteristics of fluids:

The following fluid characteristics, are relevant in technical hydraulics:

- Density
- Viscosity (tenacity)
- Surface tension
- Thermal expansion
- Compressibility, Elasticity
- Vapour pressure
- Specific heat
1.1.3.1 Density

The density of a liquid is important for the computation of

- Hydrostatic pressure ratios and forces
- Inertial forces

The density of homogenous matter is defined as the quotient of mass $M$ and volume $V$. Specific weight $\gamma$ is the product of density $\rho$ and the acceleration due to gravity $g$.

\[ \rho = \frac{M}{V} \text{[kg/m}^3]\]

\[ \gamma = \rho \cdot g \text{[N/m}^3]\]  (1.8)

1.1.3.2 Viscosity

Viscosity is a measure of the internal friction of a fluid. Each fluid has a resistance against its deformation, which occurs due to the molecular impulse exchange inside the fluid. This can be described as follows:

The molecules in two neighbouring fluid layers move in the same direction (Fig. 1.6). Between the fluid layers there is an imaginary separation plane. It is assumed that all molecules of the same layer move with the same velocity. The molecule velocities in two layers are different. Since the separation plane is permeable, molecule exchange between the fluid layers occur through diffusion.

![Fig. 1.6: Fluid layers with different velocities](image)

Molecules which move from the faster layer to the slower layer transfer a velocity surplus and thus an impulse surplus (impulse = mass times velocity difference). During the transition from the slower to the faster layer they transfer an impulse deficit.

An impulse surplus produces an accelerating force and an impulse deficit a retarding force in the fluid layer (force = impulse per time). The deceleration and acceleration forces are the cause of the shear stress arising in the separation plane between the fluid layers (tension = stress per area).
The deformation of a fluid results in velocity difference between the fluid layers (Fig. 1.5). The described shear stress works against this velocity difference. Their size determines the viscosity (tenacity) of a fluid.

The proportionality factor $\mu$, contained in Newton's hypothesis, is the dynamic viscosity with the unit [Pa s]. Dynamic viscosity in relation to density gives the

\[ \nu = \frac{\mu}{\rho} \]  

(1.9)

Besides the pure Newton's fluids where the shear stress is proportional to the deformation rate $\frac{du}{dt}$ (or velocity gradient $\frac{dv}{dy}$), there are other fluids this relation does not apply to. The following rheologic diagram represents the relationship between shear stress and velocity gradient of different fluids.

![Rheologic diagram](image)

**Fig. 1.7: Rheologic diagram**

**Newton's fluids**

They correspond to the Newton's hypothesis (see formula 1.7). Due to the direct proportionality between shear stress and velocity gradient, Newton's fluids are illustrated as straight lines in the rheologic diagram.

The gradient of the straight line corresponds to the dynamic viscosity of the fluid. Therefore a strongly viscous fluid is represented by a steep straight line and a little viscous fluid is represented by a flatter straight line. Examples for Newton's fluids are: Water, organic solvents, liquid hydrocarbons, oils, air.
**Non-Newtonian fluids**
They are characterised by a nonlinear connection between shear stress and velocity gradient. The shear stresses, which are needed to deform the fluid, are larger with small deformation rates than with large deformation rates. This behavior is typical in suspensions. Once brought into motion, small shear stresses are enough to keep them in motion.
Examples: Bentonit suspension, liquid concrete

**Bingham fluids**
They show plastic behaviour. Below a critical shear stress \( \tau_0 \) they are flexible like a solid body and above \( \tau_0 \) they behave like a Newton fluid.

\[
\tau_{xy} = \tau_0 + \mu \cdot \frac{dv}{dy}
\]

(1.10)

All solids show the characteristics of a Bingham fluid in the transition to the plastic range.

**Ideal fluids**
The so-called ideal liquid is free of viscosity (not tough). It opposes thus no resistance to deformation. In the rheologic diagram (see Fig. 1,7) this behaviour is represented by a straight line on the horizontal axis.

**Ideal solid state**
The ideal solid body behaves in the opposite way. The straight line, which describes its behaviour, coincides with the vertical axis of the rheologic diagram.

Only Newton's fluids are regarded by the following considerations.
1.2 Types and conditions of flow

1.2.1 Pipe and channel flow

The directed movement of fluids is possible in systems with:
   1. closed cross section (pipe flow)
   2. free surface (open channel flow)

In the case of the pipe flow the fluid is enclosed generally by a pressure resistant bound. The pressure head of the fluid can be determined by piezometer.

The open channel flow is characterized by a combination from firm and fluid bound. The latter is to be understood as interface between two fluids of different density.

The most important part of the channel current in practice is the **water flow on the earth's surface**. For this special case, the characteristic is the fluid interface water/air (liquid/gaseous) at the water surface. It is essential that the water pressure at the water surface is identical with the atmospheric pressure. Because of the small changes of the atmospheric pressure in the comparison with the pressure changes in the water, the water surface can be regarded as surface of constant pressure. It is accepted generally as reference pressure and set to zero.

Fig. 1.8: Flow in closed cross section and with free surface

a) Pipe flow

b) Open channel flow
Laminar and Turbulent Flow

In 1883 Osborne Reynolds notices at his famous “Reynouls’s color-thread-experiment”, that in one pipe two different flow conditions can occur.

Water flows from a tank through a glass pipe. In order to avoid disturbances by filling the container, Reynolds had already filled it on the day before with water, so that up to the beginning of the experiment all water movements in the container had faded away. To avoid the disturbances at the pipe intake, it had been carefully expenditure-rounded. Reynolds added colored water with a cannula to the water flowing in the pipe.

1. With the first condition the molecules flow into layers and mix themselves only due to the molecular diffusion and not due to the flow movement. This condition is called laminar (Latin: lamina – the plate)

2. With the second condition the arranged current is overlaid by an irregular fluctuation movement, which leads to a mixture of the molecules. This condition is called turbulent (Latin: turbulentus – disturbed)
Systematic investigations showed that the flow condition is dependent essentially on the following three parameters:

- velocity $v$
- pipe diameter $D$
- the kinematic viscosity of the fluid $\nu$.

The following formula shows the so-called Reynolds number for the determination of the flow condition:

$$Re = \frac{v \cdot D}{\nu} = \frac{v \cdot D \cdot \rho}{\mu}$$

(1.11)

According to whether the Reynolds number of a regarded pipe flow is larger or smaller than the critical parameter $Re_{krit}$, the current can be assigned to the laminar flow, turbulent flow or the transition area between both flowing conditions.

For the pipe flow a critical Reynolds number is accepted by $Re_{krit} = 2300$.

At values below the critical Reynolds number, into the current induced disturbances are absorbed by the viscosity (tenacity) of the liquid (eliminated). Above this number the inertial forces of the molecules are larger than the viscosity forces of the fluid. Absorption of disturbances does not take place. The consequence is a turbulent current.

One deals in detail in chapter 2.1.2.2.2 about laminar and turbulent flow.

### 1.2.2 Subcritical and supercritical flow

In open channels or partly filled pipes fluid flows with a free surface (open channel flow). (Pressure-) disturbances are expressed in the form of undulations at the surface. If the regarded current has a flow velocity $v$, which is smaller than the propagation velocity $c_0$ of a shallow water wave, then the disturbance can affect both downstream and upstream. This fluid movement is called subcritical flow.

If $v$ is larger than $c_0$, then the disturbance can not spread upstream. This movement is described as a supercritical flow.

For the distinction of the two fluid movements the Froude number is used:

$$Fr = \frac{v}{c_0}$$

(1.12)

with

- subcritical flow $Fr < 1$
- supercritical $Fr > 1$
During the transition between the two fluid movements the Froude number has the value of one ($Fr=1$). The flow velocity at this transition is named as critical flow velocity $v_{crit}$ or border velocity $v_{gr}$.

The propagation velocity $c_0$ of a shallow water wave can be determined with the following formula:

$$c_0 = \sqrt{gh_m}$$  \hspace{1cm} (1.13)

Thus the following formula results for the Froude number:

$$Fr = \frac{v_m}{c_0} = \frac{v_m}{\sqrt{gh_m}} = \frac{v_m}{\sqrt{gh_m}} = \frac{v_m}{\sqrt{gh_m}}$$  \hspace{1cm} (1.14)

$$h_m = \frac{A}{b_{sp}}$$

$v_m$ – average flow velocity

$h_m$ – average flow depth

$A$ – surface flow cross section

$b_{sp}$ – water level width

<table>
<thead>
<tr>
<th>Standing water</th>
<th>Subcritical discharge</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v = 0$</td>
<td>$0 &lt; v &lt; c = v_{gr}$</td>
</tr>
<tr>
<td>$Fr = 0$</td>
<td>$Fr = v/c &lt; 1$</td>
</tr>
</tbody>
</table>
1.2.3 Steady and Unsteady Flow

A flow movement continuous over the time called steady. It applies
\[
\frac{dv}{dt} = 0
\]  
(1.15)

The flow rate or flow (see chapter 1.3.1.3) is likewise constant over the time:
\[
\frac{dQ}{dt} = 0
\]  
(1.16)

Non steady flow processes are caused by changes of the discharge over the time. They are called unsteady. That means:
\[
\frac{dv}{dt} \neq 0 \quad \text{and} \quad \frac{dQ}{dt} \neq 0
\]  
(1.17)

Unsteady processes appear in rivers as consequence of flood waves or in estuaries as tidal waves. In addition there are also artificially produced unsteady processes, e.g. the opening or near the weirs or catches as well as the pressure surge in pipes.

Steady flow has the most practical importance. It is therefore treated in particular in the course.
1.2.3.1 Steady uniform

The flow with unchanged movement is called uniform. This is possible only with constant cross sections, discharge and incline gradients. That means:

\[
\frac{dv}{dx} = 0 \quad \text{and} \quad \frac{dQ}{dx} = 0 \tag{1.18}
\]

1.2.3.2 Steady non-uniform

If the mentioned boundary conditions are not fulfilled, a non-uniform flow adjusts itself. If this is caused by a change of discharge, one differentiates with respect to collecting or distribution strains:

Collect strain: \[ \frac{dQ}{dx} > 0 \Rightarrow \frac{dv}{dx} > 0 \tag{1.19} \]

Distribute strain: \[ \frac{dQ}{dx} < 0 \Rightarrow \frac{dv}{dx} < 0 \tag{1.20} \]

If the non-uniform flow movement results from a modification of the cross-section, then the flow movement experiences one:

deceleration: \[ \frac{dQ}{dx} = 0; \ \frac{dA}{dx} > 0 \Rightarrow \frac{dv}{dx} < 0 \tag{1.21} \]

or acceleration: \[ \frac{dQ}{dx} = 0; \ \frac{dA}{dx} < 0 \Rightarrow \frac{dv}{dx} > 0 \tag{1.22} \]
1.3 Fundamentals of calculation

1.3.1 Concepts

1.3.1.1 Fluid particle

Talking about fluid particles, then a defined fluid mass with the following definitions is to understand. On the one hand the fluid mass has to be small enough in order to be able to apply to it the term of the differential, on the other hand must be so large compared with the fluid molecules that it represents the average value of all processes taking place within the molecular range.

1.3.1.2 Streamline, Path line, Stream tube, Filament

A streamline is an imaginary line, which is formed at a certain time by the velocity vectors of all fluid particles moving on this line so that the tangents of the line are identical to the velocity vectors (Fig. 1.12).

![Streamlines](image)

Fig 1.13: Stream lines

A path line is the line, which actually puts a fluid particle back in the course of the time (Fig. 1.13)
1.3.1.3 Volumetric Flow Rate resp. Discharge

In the following the steady fluid movement in a stream tube is regarded. A fluid particle, which passes through the cross-section area dA, travels the way ds in the time dt.

\[ ds = v \cdot dt \]

The fluid particles passing through the cross-section area, in time dt form a fluid volume dV. The following special case is regarded:

1. The flow direction all fluid particles is orthogonal to the surface dA
2. The velocity of all fluid particles is equal

Then the fluid volume results as:
\[ dV = dA \cdot ds = dA \cdot v \cdot dt \]

The passing fluid volumes \( dV \) through the cross section \( dA \) during the time \( dt \) refers as the flow rate or discharge.

\[ \frac{dQ}{dt} = \frac{dA \cdot ds}{dt} \]

\[ dQ = dA \cdot v \quad (1.23) \]

In a real cross section the specified boundary conditions mentioned before are not kept strictly. If the 1. condition does not apply, only the orthogonal part of the flow rate can be considered for the computation of the flow (the normal projection of the velocity vector \( v \) on the surface \( n \)). Mathematically this is the scalar product between the velocity vector \( v \) and the normal vector \( n \) (Fig 1.15).

\[ dQ = dA \cdot v \]

\[ dQ = dA \cdot (n \cdot v) \]

![Fig. 1.15: Discharge through any cross section of a streamtube](image)

The 2. condition is also not complied in real cross sections. The velocity of the fluid particles varies over the cross section. A generally accepted formula that determines the discharge \( Q \) with consideration of the velocity distribution is:

\[ Q = \int_{A} \mathbf{v}(y,z) \cdot \mathbf{n} \, dA \quad (1.24) \]

In a simplified formula (1.23) discharge is calculated by using the velocity of the flow averaged over the cross section.
1.3.2 Conservation of mass equation (The Continuity-Equation)

The continuity equation is an equation of the conservation of mass of fluid flows. The basic idea of this fundamental law of nature consists on the fact that the change of the fluid mass in a control volume passed by the fluid must be equal to the difference between the entering and the leaking fluid mass concerning this control volume.

There are numerous specialized books that discuss the three-dimensional continuity derivation, but in this course only the one-dimensional case will be discussed. The derivation of the continuity equation is derived for one dimensional steady flow in a streamtube.

![Fig. 1.16: Streamtube (Continuity)](image)

In steady flow the fluid characteristics do not depend on the time. Therefore applies:

\[
(m_l + m_R)_t = (m_R + m_O)_{t+dt}
\]

The volumes of the regarded ranges I and O can be expressed in the form \(A_1ds_1\) and \(A_2ds_2\). Thus:
\[
\begin{align*}
(m)_1 &= \rho_1 \cdot V_1 = \rho_2 \cdot V_2 = (m_0)_1 \cdot \frac{d}{dt} \\
(m)_2 &= \rho_1 \cdot A_1 \cdot ds_1 = \rho_2 \cdot A_2 \cdot ds_2 = (m_0)_2 \cdot \frac{d}{dt}
\end{align*}
\]

and divided by \( dt \):

\[
\rho_1 \cdot A_1 \cdot \frac{ds_1}{dt} = \rho_2 \cdot A_2 \cdot \frac{ds_2}{dt} \tag{1.25}
\]

The result of the derivation is the **one dimensional continuity equation**

\[
\rho_1 \cdot A_1 \cdot v_1 = \rho_2 \cdot A_2 \cdot v_2 \tag{1.26}
\]

For incompressible fluids \((\rho = \text{const.})\) and steady flow the formula (1.27) can be derived with (1.24) and (1.26), which is frequently used in practice.

\[
Q = A_1 \cdot v_1 = A_2 \cdot v_2 = \text{konst.} \tag{1.27}
\]

This formula applies to a streamtube. It can be applied also to streamtube systems (see Fig 1.17):

\[
Q = A_1 \cdot v_1 = A_2 \cdot v_2 + A_3 \cdot v_3 \tag{1.28}
\]

---

**1.3.3 Energy theorem (Bernoulli’s Equation)**

Generally, the energy theorem (or principle of conservation of energy) means:

“The total energy of a body that is neither supplied nor extracted from outside with energy is constant. Energy can occur in different forms. Within the body the transformation of forms of energy is possible”. Certainly, this applies also to fluid flows.

For the following derivation a fluid body of a streamtube is selected (Fig 1.18):
The forces affecting an acceleration to the fluid body are:

1. Resulting pressure forces on the face surface:
   - left side: \( F_{pl} = p \cdot dA \)
   - right side: \( F_{pr} = (p + dp) \cdot dA \)

   **Resulting force:**
   \[
   dF_p = -dp \cdot dA \quad (1.29)
   \]

   (The pressure forces on the boundary surface have no effect on the acceleration of the body)

2. The component of weight \( dG \) in flow direction of motion \( dG_{flow} \)

   \[
   dG = (dW) = \frac{\rho \cdot g \cdot dV}{\rho \cdot g \cdot ds \cdot dA} = \frac{dG_{flow}}{ds}
   \]

   
   with

   \[
   dG_{flow} = -\rho \cdot g \cdot dz \cdot dA \quad (1.30)
   \]

The differential fluid mass, which is accelerated by the indicated forces is \( dm = \rho \cdot dV = \rho \cdot ds \cdot dA \). With the application of the second Newton's law

\[
F = dm \cdot a
\]

the sum of the forces (in flow direction) can be indicated on the fluid body as follows:
\[ \Sigma F = 0 \]
\[ \Sigma F = dF_p + dG_{\text{flow}} = dm \cdot a \]
\[-dp \cdot dA - \rho \cdot g \cdot dz \cdot dA = \rho \cdot ds \cdot dA \cdot a \]

with \( a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{dv}{ds} \cdot v \)

\[-dp \cdot dA - \rho \cdot g \cdot dz \cdot dA = (\rho \cdot ds \cdot dA) \cdot v \cdot \frac{dv}{ds} \]

dividing by \( \rho \cdot dA \)

\[ \frac{dp}{\rho} + v \cdot dv + g \cdot dz = 0 \] (1.31)

This formula was deduced by Leonhard Euler (1750) and therefore it is called Euler-Equation:

Dividing by \( g \):

\[ \frac{dp}{\rho \cdot g} + d\left(\frac{v^2}{2 \cdot g}\right) + dz = 0 \]

\[ d\left(\frac{p}{\rho \cdot g} + \frac{v^2}{2 \cdot g} + z\right) = 0 \]

For incompressible and constant-density fluids, the one dimensional Euler-Equation can be easily integrated between two points:

\[ \frac{p_1}{\rho \cdot g} + \frac{v_1^2}{2 \cdot g} + z_1 = \frac{p_2}{\rho \cdot g} + \frac{v_2^2}{2 \cdot g} + z_2 = H \] (1.32)

This equation is called Bernoulli's equation for the ideal fluid flow. \( H \) is the energy-head of any point in the flow. It consists of the kinetic energy-head (velocity head \( v^2/2g \)) and the potential energy (pressure head \( p/\rho g \) and the height \( z \) above the datum). The potential energy-head is also called piezometric height. For ideal fluids \( H \) has the same size at all points of the current.

In ideal fluid flow is understood as a frictionless current - the viscosity remains unconsidered. In some practical applications, a frictionless current can be assumed, in particular with small cross-sections and slow laminar currents.

Real fluid flows (Newton's currents) are friction-affected. The energy-head \( H \) related to a fixed datum becomes smaller in the direction of flow due to the energy loss by friction (Fig 1.19). The energy loss is summarized in the head loss \( hv \).
\[ \frac{p_1}{\rho \cdot g} + \frac{v_1^2}{2 \cdot g} + z_1 = \frac{p_2}{\rho \cdot g} + \frac{v_2^2}{2 \cdot g} + z_2 + h_v = H \] (1.33)

Fig. 1.19: Energy heights in pipe and open channel flows

1.3.4 The momentum theorem

To develop the work-energy principle we will use a control volume which coincides with the walls of a streamtube (Fig.1.20). The second Newton's law can be applied to this control volume:

\[ \sum F = m \cdot a \]

\[ \sum F = \frac{d}{dt} (m \cdot v) \] (1.34)

The mass m is constant according to the solid state mechanics. It applies:

\[ \sum F = m \cdot \frac{dv}{dt} \]

The regarded frictionless flow in the fluid body cannot proceed with temporally constant mass. Here applies the general formulation:

\[ \sum F = \frac{dm}{dt} \cdot v + m \cdot \frac{dv}{dt} \] (1.35)
In the special case of steady flow $\frac{dv}{dt}$ is $= 0$. Then for the fluid body applies:

$$\sum F = \frac{dm}{dt} \cdot v$$

$$\sum F = \rho \cdot \frac{dV}{dt} \cdot v$$

(1.36)

The transition of the infinitesimal small fluid body with the mass $dm$ takes place to a control volume $CV$ lying in the streamtube (Fig 1.20):

$$\sum F_{CV} = \int dCV (\rho \cdot v \cdot dV) - \int dCV \rho \cdot v \cdot dV$$

(1.37)

With application of the Reynolds transportation theorem, which is not described in detail this course (for more details read recommended literature), the transition from the volume to the area integral takes place:

$$\frac{d}{dt} \int dCV \rho \cdot v \cdot dV = \int dCO_{out} \rho \cdot v \cdot (v \cdot dA) - \int dCO_{in} \rho \cdot v \cdot (v \cdot dA)$$

(1.38)

In this formula $CO$ is the surface of the control volume. The control volume fills the current tube completely. Since over the surface of a streamtube no exchange of material takes place, only the face surfaces 1 and 2 of the control volume must be regarded with the further balance. In the represented special case the flow rate is constant over the face surfaces. The velocity vectors are orthogonal to the face surfaces. With the aforementioned assumption follows after integration of the right equation side:
\[
\frac{d}{dt} \int_{CV} \rho \cdot v \cdot dV = \rho_2 \cdot v_2 \cdot Q_2 - \rho_1 \cdot v_1 \cdot Q_1
\]  
(1.39)

And with the continuity equation:
\[
\frac{d}{dt} \int_{CV} \rho \cdot v \cdot dV = \rho \cdot Q \cdot (v_2 - v_1)
\]  
(1.40)

And therefore:
\[
\sum F_{CV} = \rho \cdot Q \cdot (v_2 - v_1)
\]  
(1.41)

This is the sum of the forces at the CV only due to the change of momentum.

Here are some examples for the application of the momentum theorem for computations in hydraulic engineering:
- Derivation of the impact pressure loss after Borda Carnot
- Computation of the reactions (bearing forces) at a pipe elbow union

Fig. 1.21 Forces at a pipe elbow
2 Fluid flow at body surfaces

The flow of a real fluid around an immersed body exerts a force on its surface. This force is called drag force (or profile drag) and consists of two components:

- Friction drag due to tangential stress at the body surface (shear drag, viscous drag)
- Form drag (pressure drag) due to stress normal to the body surface

<table>
<thead>
<tr>
<th>Body shape</th>
<th>Form drag</th>
<th>Friction drag</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long body shapes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Flat plate parallel to flow</td>
<td>0%</td>
<td>100%</td>
</tr>
<tr>
<td>Streamlined shapes</td>
<td>ca. 10%</td>
<td>ca. 90%</td>
</tr>
<tr>
<td>Solid body shapes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Circular cylinder</td>
<td>ca. 90%</td>
<td>ca. 10%</td>
</tr>
<tr>
<td>Flat plate perpendicular to flow</td>
<td>100%</td>
<td>0%</td>
</tr>
</tbody>
</table>

Fig. 2.1: Influence of the body shape on the pressure and friction components of profile drag

Very significant for the formation of flow conditions at boundaries is the so-called no-slip condition. It states that the velocity near the wall approaches zero because of the influence of viscosity if the wall itself is in rest. Due to this condition the molecular shear stress is linearly
dependent on the velocity gradient (Newton’s law, formula (1.7)). An unsteadiness in the velocity profile would thus lead to the occurrence of infinite shear stresses, which is physically impossible, so the velocity has to decrease to zero in approximation of the wall, independently of its original value. This is called no-slip condition of the fluid at the wall.

2.1 Flow over flat plates

This chapter deals with friction resistance and the associated velocity distribution for flow at flat plates.

2.1.1 Laminar uniform flow

Assumption: A fluid between two plates can be set in motion by different external influences:
1. Movement of both plates relative to each other in their respective planes (Couette flow)
2. Difference of pressure in the fluid between the plates (Poiseuille flow)
If the magnitude and direction of velocity in a flow do not change at different points of time it is called uniform flow.

2.1.1.1 Couette flow (without pressure gradient)

The fluid flow is caused only by a movement of the plates relative to each other (Fig 2.2). The definition of general Couette flow with pressure gradient sometimes used in literature is different from the one presented here.

Application of the momentum theorem to a fluid particle in the case of a steady, uniform flow leads to:

\[-\frac{dp}{dx} \cdot dz + \frac{d\tau}{dz} \cdot dx = 0\]

(2.1)
In the case of Couette flow the pressure between the plates is constant:
\[
\frac{dp}{dx} = 0
\]
which implies
\[
\frac{d\tau}{dz} = 0
\]
This means the shear stress is constant over the cross section with \( \tau = \tau_0 \) (Fig 2.3).

The velocity distribution for laminar flow can be determined with Newton's hypothesis:
\[
\tau = \mu \cdot \frac{dv}{dz} = \tau_0
\]

\[
dv = \frac{\tau_0}{\mu} \cdot dz
\]

\[
v(z) = \int \frac{\tau_0}{\mu} \cdot dz
\]

\[
v(z) = \frac{\tau_0}{\mu} \cdot z + c
\]
The boundary condition \( v = 0 \) (no-slip condition) applies to the lower plate \( (z = 0) \). Thus the integration constant \( c \) is zero.

\[
v(z) = \frac{\tau_0}{\mu} \cdot z
\]
The velocity distribution is linear (Fig 2.3).

![Fig. 2.3: Velocity and shear stress distribution in Couette flow](image)
2.1.1.2 Plane Hagen-Poiseuille flow

Hagen-Poiseuille flow is sometimes only discussed as a special case of pipe flow in literature. The following derivation is based on plane (or two-dimensional) Hagen-Poiseuille flow:

Fluid flow between the plates is caused by a pressure difference in x-direction (pressure gradient). There is no movement of the plates relative to each other (Fig. 2.4).

![Fig. 2.4: Fluid particles in Hagen-Poiseuille Flow](image)

As for Couette flow the application of the momentum theorem at the fluid particle in the case of a steady uniform flow results in:

\[-\frac{dp}{dx} \cdot dx \cdot dz + \frac{d\tau}{dz} \cdot dx = 0\]  

(2.4)

It follows that

\[\frac{d\tau}{dz} = \frac{dp}{dx}\]

Replacing \(\tau\) by Newton's hypothesis \(\tau = -\mu \cdot \frac{dv}{dz}\) yields:

\[\mu \cdot \frac{d^2v}{dz^2} = -\frac{dp}{dx}\]  

(2.5)

\[d^2v = -\frac{dp}{\mu \cdot dx} \cdot dz^2\]  

(2.6)

Double integration is necessary.
\[ \frac{dv}{dx} = \left[ -\frac{dp}{\mu \cdot dx} + c_1 \right] dz \] (2.7)

\[ v = \left[ -\frac{dp}{2\mu \cdot dx} \cdot \left( b \cdot z - z^2 \right) + c_1 \cdot z + c_2 \right] \]

The integration constants \( c_1 \) and \( c_2 \) are determined with consideration of the boundary conditions (no-slip condition: \( z=0 \Rightarrow v=0 \) and \( z=b \Rightarrow v=0 \))

\[ v(z) = \frac{dp}{2\mu \cdot dx} \cdot (b \cdot z - z^2) \] (2.8)

This formula shows a parabolic velocity distribution in Hagen-Poiseuille flow (Fig. 2.5 and exercise 1).

The pressure gradient \( \frac{dp}{dx} \) can be depicted with the help of Bernoulli's equation. The equation is set up for two cross sections (the ends of the fluid body):

\[ \frac{p_1}{\rho \cdot g} + \frac{v_1^2}{2 \cdot g} + z_1 = \frac{p_2}{\rho \cdot g} + \frac{v_2^2}{2 \cdot g} + z_2 + h_V = H \]

In the regarded special case (Fig. 2.4) is \( z_1 = z_2 \) and \( v_1 = v_2 \) (continuity law with constant plate distance \( b \)). Then the head loss \( h_V \) only depends on the pressures \( p_1 \) and \( p_2 \):

\[ \frac{p_1 - p_2}{\gamma} = h_V \]

The downward gradient of the energy line \( I_E \) is the quotient of the head loss \( h_V \) and the distance \( s \) of the cross sections:

\[ I_E = \frac{h_V}{s} = \frac{dp}{\gamma \cdot s} = \frac{dp}{\gamma \cdot dx} \]

Thus formula (2.8) can be converted to the following more graphical form:

\[ v(z) = \frac{I_E \cdot \gamma}{2\mu} \cdot (b \cdot z - z^2 ) \] (2.9)

The distribution of the shear stress \( \tau \) for the laminar flow can be determined using Newton's hypothesis in formula (2.7):

\[ dv = \left[ -\frac{dp}{\mu \cdot dx} + c_1 \right] dz \]

\[ dv = \frac{dp}{\mu \cdot dx} \cdot \left( \frac{b}{2} - z \right) dz \]

\[ \frac{dv}{dz} = \frac{dp}{\mu \cdot dx} \cdot \left( \frac{b}{2} - z \right) \]

\[ \tau = \frac{dp}{dx} \cdot \left( \frac{b}{2} - z \right) \] (2.10)
This formula states that in laminar flow the shear stress $\tau$ is linearly distributed over the cross section (Fig 2.5).

![Velocity and shear stress distribution in Hagen-Poiseuille Flow](image)

2.1.2 Boundary layer flow (non-uniform flow)

Laminar flow is a special case that rarely occurs in civil engineering practice. Most of the regarded flow types are turbulent. Below is a closer look at the transition zone between a turbulent flow and the no-slip condition at a boundary as well as the flow conditions that prevail there.

Contrary to the laminar flow regarded in chapter 2.1.1 the transition zone of a turbulent flow is not homogeneous, i.e. the transition zone changes in flow direction.

Between the free turbulent flow and the boundary a so-called boundary layer is formed. Within this boundary layer the flow is real (viscous). Outside the boundary layer the so-called free stream is assumed to be ideal (inviscid).

In illustration 2.6 free turbulent flow with velocity $v_0$ parallel to a flat plate is shown (side-view). At the plate the no-slip condition applies. A boundary layer is formed in the transition zone between plate and free flow. It is characterized by a velocity distribution $v(z)$, where $z$ is the vertical distance from the plate. The boundary layer thickness $\delta$ increases along the plate since more fluid particles are decelerated. The beginning of the boundary layer was arbitrarily defined, since there is no clear border between boundary layer and outer flow. It is the place where the the local velocity $v(z)$ differs from the free stream velocity $v_0$ by 1%.
The boundary layer flow, like all real flows, can be laminar or turbulent. At the beginning of the plate the boundary layer thickness $\delta$ is still very small and the boundary layer flow in $z$-direction is limited, so a laminar boundary layer is formed. With increasing boundary layer thickness the possibility of motion in $z$-direction of the fluid particles continues to increase. A fluctuation perpendicular to the main flow direction takes place. The laminar layered flow dissolves and the flow becomes turbulent. The zone between both kinds of flow is called transition zone. In illustration 2.6 the time-averaged velocity profiles $\nu(z)$ and the value of the respective wall shear stresses $\tau_0$ are represented. Local Reynolds numbers are defined for distinction of the different kinds of boundary layer flow:

As a function of the thickness $\delta$:

$$\text{Re}_\delta = \frac{v_0 \cdot \delta}{\nu}$$ (2.12)

or: Depending on the distance $x$ of the edge of the plate:

$$\text{Re}_x = \frac{v_0 \cdot x}{\nu}$$ (2.13)

### 2.1.2.1 Laminar boundary layer

For the laminar boundary layer Blasius (1908) made a detailed flow analysis. Its result is a formula for determining the boundary layer thickness $\delta_{\text{lam}}$ along the plate:

$$\delta_{\text{lam}} = 5,0 \sqrt{\frac{\nu \cdot x}{v_0}} = \frac{5,0x}{\sqrt{\text{Re}_x}}$$ (2.14)
The thickness of the boundary layer is therefore proportional to the root of the distance $x$ of the edge of the plate. This formula meets the mentioned requirement that the boundary layer begins where the local velocity differs by 1% from the free stream velocity.

The laminar boundary layer flow resembles the Hagen-Poiseuille flow described in chapter 2.1.1.2 (laminar, with pressure gradient). The **velocity distribution** in the laminar boundary layer is almost **parabolic**. This was predicted by Blasius and confirmed in numerous experiments:

$$v_x(z) = \frac{v_0}{\delta^2} \left(2 \cdot \delta \cdot z - z^2\right)$$  

(2.15)

The value of the wall shear stress $\tau_0$ can be determined using the following formula (for proof see Street [5] chap. 7.5):

$$\tau_0 \cong \frac{\rho \cdot v_0^2}{2} \cdot \frac{8}{\sqrt{15 \cdot Re_x}}$$  

(2.16)

This formula shows that the wall shear stress $\tau_0$ is inversely proportional to $\sqrt{x}$. Hence the wall shear stress decreases along the plate in direction of flow $x$ and with increasing thickness of the boundary layer $\delta$. Finally in the transition zone the shear stress increases again. The **shear stress distribution** in $z$-direction is **almost linear**.

**2.1.2.2  Turbulent boundary layer**

The turbulent boundary layer is of a more complex structure than the laminar boundary layer. Near the boundary a very thin **viscous sublayer** (characterized by its tenacity) is formed. The overlying turbulent zone can be divided into two layers (Fig. 2.7). Between viscous sublayer and outer turbulent zone a turbulent transition zone is located, in which the viscous force still noticeably but not yet completely absorbs the turbulence.
2.1.2.2 Viscous sublayer

Sometimes the viscous sublayer is also called laminar boundary layer in literature. No real layered flow is formed because of interfering fluid movement in the adjacent turbulent boundary layer. The viscous sublayer is a wavy flow with dominating viscous forces.

The viscous sublayer flow corresponds approximately to the Hagen-Poiseuille flow (laminar, with pressure gradient). As shown in chapter 2.1.1.2, the velocity distribution is parabolic (formula (2.8)) and the shear stress distribution is linear (formulas (2.10), (2.11)).

The formation of the viscous sublayer depends mostly on the surface texture (roughness) of the boundary. One differentiates between the following roughnesses:

- hydrodynamically smooth
• transitional behaviour
• hydrodynamically rough
• hydrodynamically extremely rough

The flow behaviour and the structure of a turbulent boundary layer as a function of different roughnesses will be dealt with in detail in chapter 3.4.2.2.

2.1.2.2.2 Turbulent layer

This chapter continues to explain the differences between laminar and turbulent flow.

Under **laminar flow** conditions the fluid particles slide past each other in layers of different velocity, which results in molecular momentum exchange causing viscous friction forces and viscous shear stress (see chapter 1.1.3.2). Newton’s friction hypothesis states that shear stress and velocity are directly proportional to each other.

Under turbulent flow conditions, in contrast to laminar flow, the fluid particles take irregular courses. Looking at a fixed place in the flow the passing fluid particles have different velocities and directions of motion.

Illustration 2.9 shows the absolute velocities in x-direction (main flow direction) and in z-direction (perpendicular to the main flow) at a fixed place for different points of time.

![Illustration 2.9: velocity fluctuations in turbulent flow](image)

**Fig. 2.9: velocity fluctuations in turbulent flow**

When looking in a macroscopic way at fluid particles it can be observed that so-called turbulence “packages” shift perpendicular to the main flow direction, so fast flowing fluid particles from the free stream that move closer to the boundary have to be decelerated. In
response, slowly flowing fluid masses in proximity of the boundary must be accelerated when moving further away from it. This normal movement of turbulence packages causes continuous momentum exchange between inner and outer zone. The arising deceleration and acceleration forces (momentum exchange of fluid parcels) cause so-called “apparent turbulent shear stress”. The magnitude of the shear stress due to turbulence exceeds the magnitude of shear stress due to viscosity (see chapter 1.1.3.2), so in turbulent flow viscous shear stress can generally be neglected.

The turbulent shear stress is determined as follows:
One defines the fluid mass exchange between close and far flow layers with the help of an imaginary control area parallel to the boundary. The main stream flows along this surface with the following instantaneous velocity (Fig. 2.9):

\[ v_x = \overline{v_x} + v'_x \]  

with \( v_x \) – instantaneous velocity in x-direction  
\( \overline{v_x} \) – time-averaged fluctuation velocity  
\( v'_x \) – fluctuation in x-direction

Note: \( \overline{v'} = 0 \) (average fluctuation velocity is zero)  

Fluid (turbulence packages) flowing through the control area with any angle \( \alpha \) possesses an instantaneous velocity \( v \), a volume \( dV \) and a mass \( dm \) (Fig. 2.10). The instantaneous flow of a fluid packages is:

\[ dm = \frac{dm}{dt} = \rho \cdot v \cdot dA \cdot \sin \alpha \]  

The instantaneous momentum flow of a fluid packages is:

\[ dI = \frac{dl}{dt} = \frac{dm \cdot v}{dt} = dm \cdot v \]  

with (2.19)  
\[ dI = \rho \cdot v \cdot v \cdot dA \cdot \sin \alpha \]  

Fig. 2.10: “apparent” shear stress
The time derivative of the momentum (instantaneous momentum flow corresponds to a force \(dF\) in direction of the instantaneous velocity \(v\) of the fluid packages. The component of \(dF\) in main stream direction \(x\) is:

\[
dF_x = \hat{d}\tau \cdot \cos \alpha = \rho \cdot v \cdot v \cdot dA \cdot \sin \alpha \cdot \cos \alpha
\]  

(2.22)

with \(v_x = v \cdot \cos \alpha\) \(v_z = v \cdot \sin \alpha\)

\[
dF_x = \rho \cdot v_x \cdot v_z \cdot dA
\]  

(2.23)

d\(F_x\) is a shearing force acting on the regarded control area \(dA\), so the instantaneous shear stress in main flow direction at the control area is:

\[
\tau = \frac{dF_x}{dA} = \rho \cdot v_x \cdot v_z
\]  

(2.24)

However the instantaneous shear stress \(\tau\) is not of interest, but the time-average determined with the help of the time-averaged velocities.

\[
\bar{\tau} = \rho \cdot \bar{v}_x \cdot \bar{v}_z
\]  

(2.25)

The value for \(v_x\) is known from formula (2.18). The instantaneous speed \(v_z\) normal to the main flow direction corresponds to the fluctuation magnitude in \(z\)-direction:

\[
v_z = v'_z
\]  

(2.26)

Formulas (2.17) and (2.26) are substituted into (2.25):

\[
\bar{\tau} = \rho \cdot (\bar{v}_x + v'_x) \cdot v'_z
\]  

(2.27)

\[
\bar{\tau} = \rho \cdot \bar{v}_x \cdot v'_z + \rho \cdot v'_x \cdot \bar{v}_z
\]

The first term on the right side of the formula is zero (2.18). The second term is the product of the average fluctuations \(v'_x\) und \(v'_z\). The product is negative (and not equal to zero), which can be explained as follows:

A turbulence packages which moves in \(z\)-direction and away from the boundary has a positive velocity \(v_z = v'_z\). However it also causes a deceleration of the flow in \(x\)-direction due to its momentum deficit. The result is a negative fluctuation \(v'_x\). If a turbulence packages moves towards the boundary, the values turn around (Fig. 2.9).

The formula for determining the time-averaged turbulent shear stress is:

\[
\bar{\tau} = \rho \cdot v'_x \cdot v'_z
\]  

(2.28)
This very good formula for turbulent shear stress was confirmed experimentally. For computations in practice this formula is not evaluable. With the help of Prandtl's mixing length theory it is possible to establish a connection between the fluctuation velocities \( v'_x, v'_z \) and the time-averaged velocity. Prandtl (1925) assumes that each turbulence packages has to travel over a length \( l \) (mixing length) before its momentum is transferred (see Fig 2.11).

**Fig. 2.11: Mixing length after Prandtl**

The change of the mean velocity \( \bar{v}_x \) as a function of the distance from the wall is:

\[
\frac{dv_x}{dz} \quad (2.29)
\]

The fluctuation \( v'_x \) is caused by a turbulence packages travelling in \( z \)-direction over the mixing length \( l \). Its magnitude is the velocity difference for the times of the formation of the packages and the momentum exchange:

\[
v'_x = -l \cdot \frac{dv_x}{dz} \quad (2.30)
\]

For continuity reasons \( v'_x \sim v'_z \) and thereby:

\[
v'_z = -l \cdot P \cdot \frac{dv_x}{dz} \quad (2.31)
\]

with \( P \) – proportionality factor
Fig. 2.12: Fluctuations $v'_x$ und $v'_z$

The proportionality factor $P$ can be included in the unknown mixing length. If (2.30) and (2.31) are substituted into (2.28), the following relationship can be seen:

$$
\tau = \rho \cdot \frac{\overline{v'_x \cdot v'_z}}{l^2} \cdot \frac{\overline{dv_x}}{dz}^2
$$

(2.32)

Investigations of Nikuradse provided the result that the mixing length in proximity to the wall is proportional to the distance from the wall $z$. This implies:

$$
l = \kappa \cdot z
$$

(2.33)

with $l$ – mixing length

$\kappa$ – „von Karman“-constant

$z$ – distance from the wall

According to Prandtl the Karman constant in wall proximity is $\kappa = 0.4$. If (2.33) is substitute into (2.32) and omits the overline for the time-average, one receives the following formula for the computation of the turbulent shear stress:

$$
\tau_{\text{turb}} = \rho \cdot \kappa^2 \cdot z^2 \cdot \left(\frac{dv_x}{dz}\right)^2
$$

(2.34)

Therefore the **turbulent shear stress is proportional to the square of flow velocity**. This differentiates laminar and turbulent flow considerably from each other.

With consideration of the viscous shear stress the complete formula for determining the shear stress arising in fluid flow is:

$$
\tau_{\text{total}} = \tau_{\text{visk}} + \tau_{\text{turb}}
$$
\[
\tau_{\text{total}} = \rho \cdot \nu \cdot \frac{dv}{dz} + \rho \cdot \kappa^2 \cdot z^2 \cdot \left(\frac{dv}{dz}\right)^2
\]  

(2.35)

with \( \rho \) – fluid density

\( \nu \) – kinematic viscosity

\( \nu \) – time-averaged velocity in x-direction

\( \kappa \) – „von Karman“-constant

\( z \) – distance from the wall

Here the shear velocity \( \nu^* \) is introduced. It is defined as follows:

\[
\nu^* = \sqrt{\frac{\tau_0}{\rho}}
\]  

(2.36)

with \( \nu^* \) – shear velocity

\( \tau_0 \) – shear stress on the wall

\( \rho \) – fluid density

In chapters 2.1.1 and 2.1.2.1 the velocity distributions over the flow cross section under laminar flow conditions were examined. In the following the velocity distribution in turbulent flow \( v_x \) as a function of the distance from the wall \( z \) is deduced. In addition the following assumptions are necessary:

- The mixing length increases linearly with the distance from the wall \( z \) (formula (2.33)). Actually this hypothesis is only correct in proximity of the wall.

- The shear stress \( \tau \) is equal to the shear stress on the wall \( \tau_0 \), and equally distributed over the flow cross section (distance from the wall \( z \)). The derivations in chapter 2.1.1 do not confirm this.

Only with the help of these assumptions the following transformation of formula (2.34) is possible:

\[
\frac{dv_x}{dz} = \sqrt{\frac{\tau_0}{\rho \cdot \kappa^2 \cdot z^2}}
\]  

(2.37)

with formula (2.36)

\[
\frac{dv_x}{dz} = \frac{\nu^*}{\kappa \cdot z}
\]  

(2.38)

Integration of the formula results in:

\[
v_x(z) = \frac{\nu^* \cdot \ln(z)}{\kappa} + C
\]  

(2.39)
This is the “von Karman-Prandtl law”. Accordingly the velocity is distributed logarithmically over the flow cross section in turbulent flow at static boundaries. The integration constant C depends only on the boundary conditions and must be determined separately for every regarded turbulent flow layer (Fig 2.7 – interior and outer zone). This will not be discussed in detail here.

So far the formulas for determining the turbulent shear stress (formula (2.34)) and the velocity distribution in turbulent flow (formula (2.39)) have been derived. Continuing chapter 2.1.2.1 the formula for determining the thickness \( \delta \) of the turbulent boundary layer is presented (no proof):

\[
\delta_{\text{turb}} = \frac{0.37 \cdot x^{4/5} \cdot \nu^{1/5}}{v_0^{1/5}} = \frac{0.37 \cdot x}{Re_x^{1/5}}
\]  

(2.40)

The formula shows that the thickness of the turbulent boundary layer in direction of flow increases much faster \((\delta \sim x^{4/5})\), than the thickness of the laminar boundary layer (formula (2.14) and Fig. 2.6).

A substitution of the boundary layer thickness \( \delta_{\text{turb}} \) (determined with formula (2.40)) into formula (2.41) yields a function for the wall shear stress \( \tau_0 \) at smooth walls.

\[
\tau_0 = 0.0225 \cdot \rho \cdot u_0^2 \cdot \left( \frac{v}{u_0 \cdot \delta} \right)^{1/4}
\]  

(2.41)

\[
\tau_0 = \frac{\rho \cdot u_0^2}{2} \cdot 0.058 \cdot Re_x^{1/5}
\]  

(2.42)

2.2 Flow at solid bodies

The flow over a flat plate described in chapter 2.1 is a special case. With this type of flow the profile drag (as presented in the introduction to chapter 2) does not consist of the two components frictional drag and form drag. In this special case only frictional drag is relevant.

For all other body shapes the consideration of the form drag force is essential for the determination of the profile drag. Frictional drag is less important depending on the shape and surface of the body.

The total drag at an (inclined) body is composed as follows (see illustration 2.13):
The uplift works perpendicular to the flow direction and is only considered in special cases. Its effect is sometimes desired, e.g. at airplane wings, but sometimes also unwanted, e.g. the “uplift” of racing cars. If a body is symmetrical relative to the flow direction the uplift is zero. For applications treated in this script $F_L$ is much smaller than $F_W$. From now on only $F_W$ in flow direction will be regarded. It is as follows (compare introduction to chapter 2):

$$F_W = F_f + F_p$$

(2.44)

with $F_W$ – drag without uplift
$F_f$ – frictional drag
$F_p$ – pressure (form) drag

### 2.2.1 Determination of frictional drag

The friction drag $F_f$ works due to the wall shear stress $\tau_0$ at the surface of an immersed body. It can be determined with formula (2.45).

$$F_f = \int_A \tau_0 \cdot t \cdot dA$$

(2.45)

with $F_f$ – frictional drag
$\tau_0$ – shear stress due to flow
$A$ – body surface area
$t$ – tangent vector on control area $dA$
Exemplary for the case of a flat plate with constant width \( B \) and length \( L \) the frictional drag \( F_f \) on one side of the plate (as in chapter 2.1) is:

\[
F_f = \int \tau_0 \cdot dA
\]

\[
F_f = B \cdot \int_0^L \tau_0 \cdot dx
\]  

(2.46)

Using formula (2.42) for determining the wall shear stress \( \tau_0 \) at smooth walls yields to:

\[
F_f = \frac{\rho \cdot v_0^2}{2} \cdot 0.073 \cdot \frac{B \cdot L}{Re_L^{1/5}}
\]  

(2.47)

with \( Re_L = \frac{v_0 \cdot L}{v} \)

Usually the frictional behavior of a body is expressed with the dimensionless friction coefficient \( c_f \). The general formula for \( c_f \) is:

\[
c_f = \frac{F_f}{\frac{\rho \cdot v_0^2}{2} \cdot A}
\]  

(2.48)

With known friction coefficient \( c_f \) of a body the friction drag \( F_f \) does not have to be determined with formula (2.45), but can more simply be obtained with formula (2.49):

\[
F_f = \frac{\rho \cdot v_0^2}{2} \cdot c_f \cdot A
\]  

(2.49)

### 2.2.2 Determination of form drag (pressure drag)

The form drag of an immersed body can be computed with the help of the following formula:

\[
F_p = \int_A -p \cdot n \cdot dA
\]

(2.50)

with \( F_p \) – form drag

\( p \) – compression stress due to flow

\( A \) – body surface area

\( n \) – vector perpendicular to control area \( dA \)

in simplified form:

\[
F_p = \frac{\rho \cdot v_0^2}{2} \cdot c_p \cdot A
\]  

(2.51)
In order to be able to compute the form drag \( F_p \), the compression stress distribution \( p \) must be known for the entire body surface area. This again depends mostly on the flow velocity distribution at the body surface. The two following chapters will give an idea of the computation of velocity and pressure distribution for the example of a circular cylinder. Furthermore the corresponding flow conditions (including boundary layer) will be dealt with.

2.2.2.1 Ideal flow around a circular cylinder

A circular cylinder with radius \( a \) is immersed in an ideal (i.e. frictionless) fluid. In Fig 2.14 the streamlines around the cylinder are shown, where the cylinder surface \( (r = a) \) is also a streamline. For the surface the velocity distribution can be determined with the following formula (derivation in the lecture):

\[
v = \left| v(a, \varphi) \right| = 2 \cdot v_0 \cdot \sin \varphi
\]  

Fig. 2.14: Ideal fluid flow around a circular cylinder

Formula (2.52) gives the result that the flow rate has a minimum \( (v_S = 0) \) at the stagnation points \( S \) and a maximum \( (v_B = 2v_0) \) at the peak \( B \). The flow velocity doubles on its travel over the cylinder surface. This velocity increase leads to a pressure decrease in the flow (see chapter 1.3.3, energy theorem (Bernoulli's equation)). Here the term pressure coefficient \( c_p \) is introduced:

Bernoulli's equation for two arbitrary points in the flow is:

\[
\frac{p_1}{\rho \cdot g} + \frac{v_1^2}{2 \cdot g} + z_1 = \frac{p_2}{\rho \cdot g} + \frac{v_2^2}{2 \cdot g} + z_2 = H
\]

The sum of pressure head \( p/\rho g \) and elevation head \( z \) is also called hydraulic or piezometric head \( h_{\text{piez}} \) (see chapter 1.3.3). Thus Bernoulli's equation for one point in the free flow \( (v_0; p_0) \) and another arbitrary point \( (v; p) \) gives:
\[ h_{\text{piez}} - h_{\text{piezo}} = \frac{v_0^2 - v^2}{2 \cdot g} \]  

(2.53)

Divided by the velocity head \( v_0^2/2g \) of the free flow:

\[ c_p = \frac{h_{\text{piez}} - h_{\text{piezo}}}{v_0^2} = 1 - \left( \frac{v}{v_0} \right)^2 \]  

(2.54)

If the elevation head change \( z \) is neglectably small compared to the pressure head change, the result is:

\[ c_p = \frac{p - p_0}{\frac{v_0^2}{2}} = 1 - \left( \frac{v}{v_0} \right)^2 \]  

(2.55)

The expression \( \rho \cdot \frac{v_0^2}{2} \) is called \textit{dynamic pressure} of a flow.

Therefore the pressure coefficient \( c_p \) is the quotient of the pressure difference of the regarded points and the dynamic pressure.

At the stagnation point \( S \) the flow velocity and thus the kinetic energy head of the flow are equal to zero. This means the entire kinetic energy has been converted to pressure energy. The pressure difference \( p - p_0 \) corresponds to the dynamic pressure. The pressure coefficient \( c_p \) for this state is equal to one. The magnitude of the pressure coefficient at the cylinder surface is shown in Fig. (2.15).

Fig. 2.15: Pressure distribution at the cylinder surface (ideal fluid)

For the case that \( c_p = 0 \) the pressure at the cylinder surface is equal to the pressure of the free flow. A negative pressure is possible.
The pressure force in direction of flow $F_{px}$ (determined by integrating the pressure distribution $p$ over the cylinder surface) is zero ($F_{px} = 0$) due to the symmetry to the ordinate. This is in contradiction to real fluid conditions in practice and is called “d’Alembert's Paradox”.

2.2.2.2 Immersion of a circular cylinder in a real fluid

In real fluids the viscosity $\nu$ is not equal to zero. The consequence is a flow pattern completely different of an ideal flow pattern. Fig. 2.16 shows the flow pattern of a circular cylinder with large Reynold's number. At the front of the cylinder a thin boundary layer with width $\delta$ is formed, which increases with increasing distance from the stagnation point S. Due to a narrower flow the velocity of the flow increases up to the widest point B of the body. At the same time the pressure of the flow decreases ($dp/ds < 0$) (Bernoulli).

![Fig. 2.16: Cylinder immersed in flow with low viscosity](image)

The pressure in the boundary layer is equal to the ambient pressure of the outer flow. The velocities of outer flow and boundary layer flow near the body are very different. The velocity in proximity of the body is very small because of the no-slip condition ($v = 0$) at the boundary. After passing the peak B, the outer flow retards due to flow "expansion". This entails a reversal of the pressure gradient ($dp/ds > 0$) (Bernoulli). As shown in chapter 2.1.1.2 (Hagen-Poiseuille flow) a fluid generally only flows in direction of a pressure gradient ($dp/ds < 0$). A flow against the pressure gradient is only possible if the momentum of the flow ($I = m \cdot v$) is big enough. This is the case in the accelerated outer flow, but in the boundary layer near the boundary the flow velocity and thus the momentum of the flow are very small. Due to the reverse pressure gradient the flow direction turns around. A backflow on the side of the cylinder is the result and the boundary layer separates from the body. A is the point of separation of the boundary layer. Due to the backflow eddies are formed behind the cylinder.

The flow behaviour at the front of the cylinder is similar for ideal and real fluids, but in the back they show very different behaviours.
The magnitude of the pressure coefficients at the cylinder surface as experimentally determined is represented in Fig. 2.17. At the back of the cylinder the pressure is negative and about constant \( (c_p = -1.2) \), which results in the thrust force \( F_{px} \). Since the pressure distribution depends a lot on the body shape, the thrust force is called form drag \( F_p \) \( (F_{px} = F_p) \).

![Pressure distribution at the cylinder surface](image)

**Fig. 2.17: Pressure distribution at the cylinder surface**

The point of separation of the boundary layer is deciding for the form drag \( F_p \): If the boundary layer separates early a broad wake zone and a large drag are the result, whereas a later separation causes the wake zone to be narrower and thus \( F_p \) to be smaller.

The influence of viscous friction on the flow behaviour has to be seen relative to the forces of inertia. The measure for this is the Reynold's number. At the cylinder is \( Re = \frac{v_o D}{\nu} \), where \( D = 2a \) is the diameter. The conditions pointed out above, leading to a separation of the boundary layer, arise with higher Reynold's numbers (i.e. with relatively low viscosity). The critical number for this is about 50. However, with a very low Reynold's number the influence of viscosity is so big that the entire flow in the body's environment is controlled by friction forces.

### 2.2.3 Profile drag of different body shapes

The profile drag is the sum of frictional drag due to the no-slip condition at the body surface and form drag due to the pressure distribution. For aerodynamic bodies (flat plate, streamlined body) the friction drag is more important, whereas for compact bodies the form drag is determining.

\[
F_W = F_f + F_p
\]

Substituting formulas (2.45) and (2.50):

\[
F_W = \int_A \tau_0 \cdot \mathbf{t} \cdot dA + \int_A \mathbf{p} \cdot \mathbf{n} \cdot dA \quad (2.56)
\]
In simplified form:

\[ F_W = \frac{\rho \cdot v_0^2}{2} \cdot c_r \cdot A_r + \frac{\rho \cdot v_0^2}{2} \cdot c_p \cdot A_p \]  
(2.57)

If the frictional drag is negligibly small \((F_f \ll F_p)\) formula (2.57) can also be written as follows:

\[ F_W = \frac{\rho \cdot v_0^2}{2} \cdot c_w \cdot A_p \]  
(2.58)

\(A_p\) is the projected cross-section area (i.e. the maximum cross-section area normal to the flow direction).

\(\text{Fig. 2.18: Cross-section areas } A_p \text{ of different bodies}\)

For determining the friction coefficient \(c_w\) of a body investigations in a flow channel are usually necessary. A simple computation is generally not possible due to the high complexity of the flow conditions.

\[ c_w = \frac{F_W}{\frac{\rho \cdot v_0^2}{2} \cdot A_p} \]  
(2.59)

Here the profile drag \(F_W\) has to be determined. This can be done directly by measurement of the actual forces on the body, or indirectly by determining the pressure conditions at the body surface. In the following two examples the procedure for determining the profile drag will be explained.

### 2.2.3.1 Flat plate perpendicular to flow direction (high Reynold’s number)

For a plate perpendicular to the flow direction (Reynolds number \(10^4\) or higher) the flow and pressure conditions are represented in fig. 2.19.

At the front side \(A_{p\text{front}}\) positive pressure (dynamic pressure) prevails. The mean pressure coefficient is \(c_{p\text{front}} \approx +0.8\) \((p_{\text{front}} > p_0)\). At the back side \(A_{p\text{back}}\) constant negative pressure \(c_{p\text{back}} \approx -1.2\) \((p_{\text{front}} < p_0)\) prevails. For the computation of the profile drag \(F_W\) formula (2.56) is used, omitting the friction drag \(F_f\):
After transforming the formula for determining the pressure coefficient \( c_p \) (formula (2.55)): \[
p = p_0 + c_p \cdot \rho \cdot \frac{v_0^2}{2}
\] and substituting into formula (2.60):

\[
F_W = \int_A -p \cdot n \cdot dA
\] (2.60)

The result of \( p_0 \) is zero and \( A_{p\text{ front}} \) is equal to \( A_{p\text{ back}} \). After substituting the measured mean pressure coefficients the profile friction coefficient \( c_W \) can be determined directly with the formula (see formula (2.58)):

\[
F_W = (0,8 + 1,2) \frac{\rho \cdot v_0^2}{2} \cdot A_p
\] (2.61)

The plate (with a Reynolds number higher than \( 10^4 \)) has a profile friction coefficient of \( c_w \approx 2,0 \). This value has been confirmed by numerous experiments in flow channels.

### 2.2.3.2 Circular cylinders with different Reynold’s numbers

In the following the profile friction coefficients \( c_W \) of circular cylinders as a function of their Reynold’s number are presented. The circular cylinder has the diameter \( D \). Incompressible flow and a hydrodynamically smooth cylinder surface is assumed (\( k_S/D = 0; k_S \) – roughness, will be dealt with in chapter 3.4.2.1). The following text and figures are partly taken from a script of the IfH (University of Karlsruhe).

\textbf{Remark 4:} With very small Reynold’s numbers laminar flow prevails. Inertia is insignificant. The \( c_W \) coefficient is strongly \( Re \)-dependent. An analytic solution according to Lamb (1932) describes this range (see fig. 2.20):
$$c_w = \frac{8\pi}{\text{Re} \cdot (2 - \ln \text{Re})} \quad (2.62)$$

**4 ≤ Re ≤ 40:** In this range inertia becomes increasingly more important. Separation zones with stationary eddies form at the back.

**40 ≤ Re ≤ 300:** In this range there is a distinct laminar boundary layer, which separates about at the widest points of the cylinder. However, a periodic interaction between the detached eddies occurs, so that in the wake a very distinct system of alternating eddies is formed. This phenomenon was first described analytically by Karman and is called Karman's vortex street. The frequency of the periodic separations can be determined with the so-called Strouhal number (Str) (Fig 2.20):

$$\text{Str} = \frac{f \cdot D}{v_0} \quad (2.63)$$

Within the range of 40 ≤ Re ≤ 300 Str increases continuously from approximately 0.16 to 0.21. This is important for technical applications, since a reaction of these flows to the body takes place, i.e. a vibration stimulation (e.g. vibrating and audible whirring of wire) can take place.

**300 ≤ Re ≤ 3 \times 10^5:** The laminar boundary layer separates near the widest points of the cylinder. The eddies formed in the wake zone are however no longer steady and arranged, but disintegrate rapidly to a turbulent wake flow with a width corresponding approximately to the cylinder diameter D. Typical $c_w$ values in this Re-range are approximately +1.2 (constant).

**Re ≤ 10^5:** For sufficiently high Reynold's numbers the boundary layer becomes turbulent. By the increased transversal momentum exchange the separation of the boundary layer and the separation points move further to the back. As a consequence a narrower turbulent wake zone is formed. The profile drag (and therefore also the $c_w$ value) becomes smaller. It can reduce approximately by a factor of 3 inside the transition zone. With even higher Re-numbers a slow increment can be registered (see fig. 2.20).
Fig. 2.20: Friction coefficients for circular cylinders

Fig. 2.21
<table>
<thead>
<tr>
<th>Body</th>
<th>$l/b$</th>
<th>$Re$</th>
<th>$C_w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular plate</td>
<td>1</td>
<td>$&gt;10^4$</td>
<td>1.18</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>$&gt;10^4$</td>
<td>1.20</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>$&gt;10^4$</td>
<td>1.30</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>$&gt;10^4$</td>
<td>1.50</td>
</tr>
<tr>
<td></td>
<td>$\infty$</td>
<td>$&gt;10^4$</td>
<td>1.38</td>
</tr>
<tr>
<td>Cylinder with flow parallel to main axis</td>
<td>0</td>
<td>$&gt;10^4$</td>
<td>0.62</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>$&gt;10^4$</td>
<td>1.15</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>$&gt;10^4$</td>
<td>0.90</td>
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<td>0.85</td>
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<tr>
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<td>0.87</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>$&gt;10^4$</td>
<td>0.99</td>
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<tr>
<td>Square bar</td>
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<td>$&gt;10^4$</td>
<td>2.00</td>
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<td>1.50</td>
</tr>
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<td>1.39</td>
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<td>Half cylinder</td>
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<td>1.20</td>
</tr>
<tr>
<td>Half cylinder</td>
<td>$\infty$</td>
<td>$&gt;10^4$</td>
<td>2.30</td>
</tr>
<tr>
<td>Half ball</td>
<td>$\infty$</td>
<td>$&gt;10^4$</td>
<td>0.39</td>
</tr>
<tr>
<td>Half ball</td>
<td>$\infty$</td>
<td>$&gt;10^4$</td>
<td>1.40</td>
</tr>
<tr>
<td>Kubus</td>
<td>$\infty$</td>
<td>$&gt;10^4$</td>
<td>1.10</td>
</tr>
<tr>
<td>Kubus</td>
<td>$\infty$</td>
<td>$&gt;10^4$</td>
<td>0.81</td>
</tr>
<tr>
<td>Konus 60°</td>
<td>$\infty$</td>
<td>$&gt;10^4$</td>
<td>0.49</td>
</tr>
<tr>
<td>Parachute</td>
<td>$\approx 3 \times 10^7$</td>
<td>$&gt;10^4$</td>
<td>1.20</td>
</tr>
</tbody>
</table>

Flow resistance factor $C_w$ for different body types (Roberson and Crowe, 1993)

Fig. 2.22
3 Open channel flow

From practical experience, the most important part of channel flow is free-surface flow. It is characterized by a separation area at the surface (water/air, liquid/gaseous), at which the water pressure is identical to the atmospheric pressure. The water surface can be regarded as a surface of constant pressure because the atmospheric pressure does not significantly change. It is generally taken as a reference pressure and defined as zero (Fig. 3.1).

Fig. 3.1: Energy levels in pipe and channel flow

Since the pressure at the surface is zero, channels can have a variable flow cross-section. In pipe flow the cross-section is fixed and the pressure changes depend on flow conditions. In channel flow on the other hand the water surface area is variable since water level and pressure line are identical. Channel flows thus have an additional degree of freedom, which makes their analysis more difficult.

3.1 Introduction

3.1.1 Channel types

We can subdivide open channels into natural or alluvial channels and artificial channels. The former are characterized by a bed formed by discharge. To understand their formation and development laws, it is necessary to have background knowledge of the hydromechanical, hydrological, geomorphologic and sedimentation processes. Artificial channels are generally
characterized by their regular geometry. Some artificial channels, for example, are navigation channels, regulated rivers, and aqueducts for water distribution systems.

The geometry of a channel (channel cross-sectional shape / channel profile) is of special importance, since the width of the wetted channel cross-section determines the amount of momentum exchange (in the form of wall shear stresses between the flow and the boundary). The term of the hydraulic radius \( r_{ny} \) is introduced here. It is defined as the quotient of the cross-sectional area of the flow \( A \) and the wetted channel perimeter \( l_U \):

\[
    r_{ny} = \frac{A}{l_U} \quad (3.1)
\]

with \( r_{ny} \) – hydraulic radius

\( A \) – cross-sectional area

\( l_U \) – wetted perimeter

Cross-sectional shapes of artificial channels

Rectangular channels:

![Fig. 3.2: Rectangular channels](image)

For rectangular channels the hydraulic radius is calculated as:

\[
    r_{ny} = \frac{A}{l_U} = \frac{h \cdot b}{2h + b} = \frac{h}{\frac{2h}{b} + 1} \quad (3.2)
\]

For very wide rectangular channels the factor \( h/b \) approaches zero. In such cases the hydraulic radius can be simplified as

\[
    r_{ny} = h. \quad (3.3)
\]
Trapezoidal channels:

![Trapezoidal Channel Diagram](image)

Fig. 3.3: Trapezoidal channels

Compound channels:

Regulated rivers are often designed as a combination of different profiles (Fig. 3.4), where the channel is divided into a main channel filled to the mean water stage (MQ), and an additional side channel, which is only used when high water (HQ) occur.

![Compound Channel Diagram](image)

Fig. 3.4: Compound channel

3.1.1.1 Natural open channel cross-sections

Natural open channel cross-sections are mostly irregular.

![Natural Open Channel Cross-Section Diagram](image)

Fig. 3.5: Natural open channel cross-section

3.1.2 Flow types and flow states in open channels

3.1.2.1 Laminar and turbulent channel flow

Laminar and turbulent flow in pipes have been dealt with in detail in chapter 1.2.2. For distinction of the flow states in open channels the dimensionless Reynolds number Re is used. Its formula for open channel flow is:
\[ \text{Re} = \frac{v \cdot 4 \cdot r_{hy}}{v} = \frac{v \cdot 4 \cdot r_{hy}}{\mu} \]  
(3.4)

with  
\( \text{Re} \) – Reynolds number  
\( v \) – mean velocity  
\( r_{hy} \) – hydraulic radius  
\( \nu \) – kinematic viscosity  
\( \mu \) – dynamic viscosity  
\( \rho \) – fluid density

For open channel flows the critical Reynolds number is \( \text{Re}_{krit} = 2300 \).

### 3.1.2.2 Subcritical and supercritical channel flow

In chapter 1.2.2 the Froude number for the distinction of subcritical and supercritical fluid movement was introduced. For preparation of chapter 3.2 the review of chapter 1.2.2 is recommended.

### 3.1.2.3 Steady and unsteady open channel flows

The terms of steady (uniform/varied) and unsteady flow are of great importance in channel hydraulics. They have been described in detail in chapter 1.2.3. In chapter 3 "Open channel flow" the special case of the steady open channel flow will be dealt with.

### 3.2 Extremal principle and specific energy

The terms subcritical and supercritical as well as critical depth \( h_{gr} \), critical velocity \( v_{gr} \) and critical slope \( I_{gr} \) are of enormous importance in the theory of open channel flow. The foundation for the following statements is the formula for the determination of the energy height \( h_{E} \) according to Bernoulli (see chapter 1.3.3):

\[ H = \frac{p}{\rho \cdot g} + \frac{v^2}{2 \cdot g} + z \]  
(3.5)

The energy height at the channel bottom \((z = 0)\) is called specific energy \( h_{E} \) of the flow:

\[ h_{E} = h + \frac{v^2}{2 \cdot g} \]  
(3.6)

Here \( h \) is the water depth. With the formula for flow \( Q \) (also called discharge in open channels hydraulics) \( Q = A \cdot v \) we get:
\[ h_E = h + \frac{Q^2}{2 \cdot g \cdot A^2} = \frac{(q \cdot b)^2}{2 \cdot g \cdot h^2 \cdot b^2} \]  

(3.7)

A is dependent on the water depth \( h \) and the channel width \( b \). In general, the width changes with the water depth. For the special case of a rectangular open channel (width \( b \) constant over water depth \( h \)) the formula can be simplified as

for rectangular channels: \[ h_E = h + \frac{q^2}{2 \cdot g \cdot h^2}. \]  

(3.8)

The relationship between \( h \) and \( h_E \) for a constant discharge \( Q \) can be represented in a diagram (Fig. 3.6). Analyzing the diagram the following is recognized:

- A minimum energy height \( h_{E,\text{min}} \) must be present in order to make the discharge \( Q \) possible. This statement is known as extremal principle.

- With an existing specific energy \( h_E \) larger than \( h_{E,\text{min}} \) there are two possibilities for the discharge.

1. Small water depth \( h \), large flow velocity \( v \); the flow is supercritical, for example in chutes, water falls or rapids. Large bottom stress is the result. Disturbances don’t spread upstream, so the computation of fluid movements always takes place downstream (in flow direction)

2. Large water depth \( h \), small velocity \( v \); the flow is subcritical. This type of movement can be found in most open channels and rivers. Disturbances like cross-section narrowing have an effect in upstream direction. Computations take place upstream.

**Fig. 3.6: h-h_E-diagram**
The extremum of equation/formula (3.8) is determined by computing the derivative $dh_E/dh$ and setting it equal to zero:

$$\frac{dh_E}{dh} = 1 - 2 \cdot \frac{q^2}{2 \cdot g \cdot h^3} = 0 \quad (3.9)$$

$$\frac{q^2}{g \cdot h_{gr}^3} = 1 \quad (3.10)$$

The water depth $h_{gr}$ is called **critical water depth**. In English publications the index $c$ for “critical” is used (in place of gr for “Grenztiefe” in German). It follows from (3.10):

for rectangular channels:  
$$h_{gr} = \sqrt[3]{\frac{q^2}{g}} \quad (3.11)$$

Substituting $q = v \cdot h$ into this formula, the **critical velocity** is:

$$v_{gr} = \sqrt{g \cdot h_{gr}} \quad (3.12)$$

By using the critical water depth $h_{gr}$ in formula 3.8 the following relationship results:

$$h_E = h_{gr} + \frac{q^2}{2 \cdot g \cdot h_{gr}^2} = h_{gr} + \frac{h_{gr}}{2}$$

For a rectangular cross-section:

$$h_E = \frac{3}{2} \cdot h_{gr} \quad (3.13)$$

This means that in the case of critical flow two third of the specific energy are present as potential energy and one third as kinetic energy. In formula (3.10) $q$ can be replaced by $q = v_{gr} \cdot h_{gr}$. It follows:

$$\frac{v_{gr}^2}{g \cdot h_{gr}} = 1$$

$$\frac{v_{gr}}{\sqrt{g \cdot h_{gr}}} = 1 = Fr \quad (3.14)$$

In the case of critical flow with the minimum energy height $h_{E,\text{min}}$ (with critical depth $h_{gr}$ and critical velocity $v_{gr}$) the Froude number is equal to one. Here the transition between subcritical and supercritical fluid movement takes place.

In literature the diagram represented in Fig. 3.6 is frequently used with exchanged $h$- and $h_E$-axes for improved clearness. Such a diagram can be seen in Fig. 3.7 for a constant discharge $q$. 

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The represented function $h_E = h_E(h)$ has an extremum for $h$ at water depth $h_{gr}$, in this case a minimum.

This means for an open channel, for example in front of an obstacle, where the specific energy $h_E$ is smaller than $h_{E,\text{min}}$, $q$ cannot be discharged and the result is backwater. The water level rises in front of the obstacle until the minimum specific energy $h_{E,\text{min}}$ is reached above the obstacle. Only then the water flows over the obstacle. However, the water level continues to rise in front of the obstacle until $q$ has been discharged completely.

It is very important to know what discharge $q$ one can get with constant specific energy height $h_E$. Fig. 3.8 shows a $q$-$h$-diagram.
For a given energy height $h_E$ a maximum discharge $q_{\text{max}}$ can take place in each cross-section. Each discharge $q < q_{\text{max}}$ can be reached with two different water depths $h$. The function $q = q(h)$ shown here exhibits an extremum for $q$ at the water depth $h_{gr}$, in this case a maximum.

It is still not known is under what boundary conditions the flow is subcritical and when it is supercritical. The slope $I$ is crucial for this. Only if the slope $I$ is larger than the critical slope $I_{gr}$, the flow is supercritical, otherwise the flow is subcritical. The critical slope is computed with the help of flow formulas. These will be dealt with in the following chapter.

3.3 Flow formulas

For the hydraulic computation of an open channel a mathematical relationship is needed – a so-called flow formula. The connection between the flow rate $v$ and the channel boundary conditions such as water depth $h$, cross-sectional characteristics $A$ and $l_U$, slope $I$ and the boundary roughness.

3.3.1 Classical flow formula after Brahms-de Chezy

The derivation of the flow formula is done assuming a steady, uniform flow. This means that two consecutive cross-sections in flow direction must have the same characteristics concerning shape and boundary roughness. Water depth $h$ and flow velocity $v$ are therefore the same for both cross-sections. The slope of the energy line $I_E$, the water surface slope $I_{Sp}$ and the bed slope $I_{Sp}$ are identical. For the derivation of the flow formula the forces applying at a water volume between the two cross-sections 1 and 2 are worked out (see Fig. 3.9).
Fig. 3.9: Forces for the derivation of the flow formula

**Driving force**: weight force in flow direction:

\[ F_G = m \cdot g \cdot \sin \alpha \]  (3.15)

**Holding force**: friction drag due to \( \tau \) at the bed:

\[ F_f = \tau_{so} \cdot l_{U,so} \cdot x \]  (3.16)

Derivation in lecture

\[ v = \frac{g \cdot A}{c \cdot l_{U,so} \cdot l_{so}} \]  (3.17)

empirical flow formula after Brahms-de Chezy

\[ v = C \cdot \sqrt{r_{hy} \cdot l} \]  (3.18)

### 3.3.2 Flow equation after Darcy-Weisbach

Replace the velocity factor \( C \) (with unit) by the hypothesis \( \sqrt{\frac{8 \cdot g}{\lambda}} \).

**general flow equation**

\[ v = \frac{1}{\sqrt{\lambda}} \cdot \sqrt{8 \cdot g \cdot r_{hy} \cdot l_E} \]  (3.19)

with  
- \( v \) – mean velocity  
- \( \lambda \) – friction coefficient  
- \( r_{hy} \) – hydraulic radius  
- \( l_E \) – (energy) slope

Derivation in lecture
With the flow equation according to Darcy-Weisbach, like with all other flow formulas, the discharge can only be approximated using assumptions and simplifications.

### 3.3.3 Manning-Strickler flow equation

#### empirical formula

According to Manning-Strickler:

\[ v = k_{St} \cdot r_{ny}^{2/3} \cdot \sqrt{E} \]  \hspace{1cm} (3.20)

**whit**
- \( v \) – mean velocity
- \( k_{St} \) – Manning-Strickler-coefficient
- \( r_{ny} \) – hydraulic radius
- \( l_E \) – (energy) slope

The Manning-Strickler formula is very common in hydraulic engineering. Due to its empirical background it is replaced increasingly by the scientifically founded flow formula by Darcy-Weisbach.

### 3.4 Friction coefficient \( \lambda \) Darcy-Weisbach - Law

#### 3.4.1 Colebrook-White’s resistance law

The Colebrook-White resistance law originally derived for the determination of the friction coefficient \( \lambda \) in pipe flow can also be applied to open channel flows under certain conditions. The law states:

\[ \frac{1}{\sqrt{\lambda}} = -2,03 \cdot \log \left( \frac{2,51}{R_e \sqrt{\lambda}} + \frac{k_s}{14,84 \cdot r_{ny}} \right) \]  \hspace{1cm} (3.21)

**with**
- \( \lambda \) – friction coefficient
- \( R_e \) – Reynolds number
- \( r_{ny} \) – hydraulic radius
- \( k_s \) – equivalent sand roughness

In the following chapters the adjustment of the law to different boundary conditions will be shown.
3.4.2 $\lambda$ depending on channel roughness

Open channel flows are fully developed turbulent boundary flows. The turbulent boundary layer reaches up to the water surface. Flow structure and velocity distribution for a hydraulically smooth channel are represented in Fig. 3.10 intersection C-D (cp. the boundary layer development at a plate chapter 2.1.2).

![Diagram of turbulent boundary layer development in a channel with ideal inlet conditions](image)

Fig. 3.10: Development of a turbulent boundary layer in a channel with ideal inlet conditions

The existing flow conditions are determined considerably by the viscous sublayer. Its formation depends on the surface roughness of the boundary and/or channel bed. The following roughnesses can be differentiated:

- hydraulically smooth
- transitional behaviour
- hydraulically rough
- hydraulically extremely rough

For further considerations it is necessary to explain the term roughness.

3.4.2.1 Roughness and equivalent sand roughness

With identical geometrical roughness height $k$, roughness elements like groove, sand grains, waves or cuboids have a different roughness depending on their arrangement (pattern), distance (density) and shape. For example plate-shaped roughness elements have the same height as the sand grains in figure 3.11, but can have a roughness of sand grains that are eight times as big depending on their distance. (Zanke [1])
Very extensive and careful systematic measurements in rough pipes were done by Nikuradse. He used circular pipes which were covered with sand of a certain grain size as densely as possible (Fig. 3.11).

The sand roughness examined by Nikuradse can be characterized by a maximum roughness density. In many technical applications the roughness density is substantially smaller. Such roughnesses can no longer be indicated by a roughness height \( k \). It has proved useful to classify roughness by comparing to a scale of a normal roughness and to choose the **Nikuradse sand roughness** for this purpose. Any roughness can generally be assigned a so-called **equivalent sand roughness** \( k_s \), which is understood as the sand grain roughness that has the same friction coefficient \( \lambda \).

The experimental determination of equivalent sand roughness \( k_s \) for a larger number of regularly arranged roughness elements has been done by Schlichting (see Fig. 3.12). (Schlichting [2])
Fig. 3.12: k – geometrical roughness height; k_s – equivalent sand roughness [2]

With the so-called natural roughnesses of commercial pipes (also called technically rough) k_s = k can be assumed in first approximation. In order to use formula (3.21) or common diagrams for determination of λ, if the roughness type is clearly different from sand roughness, k_s basically has to be computed first. Measured values from pipes or open channels with these roughness types are necessary, from which a calculation of the roughness is possible. [1] The magnitude of the equivalent sand roughness k_s can thus only be approximated for a “technically rough” boundary by “measuring” k. Otherwise it can only be determined experimentally.

<table>
<thead>
<tr>
<th>equivalent sand roughness</th>
<th>k_s in mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>plane river bed</td>
<td></td>
</tr>
<tr>
<td>– sandy or from gravel</td>
<td>d_{k, 90}</td>
</tr>
<tr>
<td>– gravel</td>
<td>50 - 100</td>
</tr>
<tr>
<td>– rough gravel</td>
<td>60 - 200</td>
</tr>
<tr>
<td>– heavy stones</td>
<td>200 - 300</td>
</tr>
<tr>
<td>– plastered floor</td>
<td>30 - 50</td>
</tr>
<tr>
<td>floodplain and bank with</td>
<td></td>
</tr>
</tbody>
</table>
In literature, e.g. Bollrich [3], and in tables, e.g. Schneider [4], the term **absolute hydraulic roughness** $k$ is used. This roughness is identical to the **equivalent sand roughness** $k_s$ described above. The symbol $k$ is generally only used for the geometrical (measurable) roughness (except for Bollrich, Schneider...).

### 3.4.2.2 Hydraulic effect of roughness

The resistance of a boundary does not only depend on the equivalent sand roughness $k_s$, but also on the characteristic cross-section dimension and shape. Such a cross-section dimension could for example be the flow depth (open channel hydraulics) or the pipe diameter (pipe hydraulics). With respect to general use the hydraulic diameter $d_{hy}$ (with $d_{hy} = 4r_{hy}$) is used as the characteristic cross-section dimension. The equivalent sand roughness $k_s$ based on the hydraulic diameter $d_{hy}$ is called **relative roughness** $k_s/d_{hy}$.

The flow resistance, characterized by the friction coefficient $\lambda$, can be very different despite the same relative roughness $k_s/4r_{hy}$. It further depends on the grade of turbulence in the
flow (intensity of the transverse movements of the fluid particles) specified by the Reynolds number Re. The Reynolds numbers occurring in natural flow conditions are much larger than the critical Reynolds number \( Re_{\text{krit}} = 2300 \). The flow in natural rivers is therefore generally turbulent.

3.4.2.2.1 Hydraulically smooth

With a relatively small (but turbulent) Reynolds number the flow in proximity of the boundary is dominated by the viscous sublayer (tenacity). If its thickness \( \delta \) (see formula (2.14)) is larger than the height of the roughness elements \( k \), the viscous sublayer covers all roughness elements and acts like a “lubricant” between the boundary and the turbulent flow. Then the roughness of the bed has no influence on the turbulent boundary layer. The flow resistance characterized by the friction coefficient \( \lambda \) only depends on the Reynolds number \( Re \).

\[
\frac{1}{\sqrt{\lambda}} = -2.03 \cdot \log \left( \frac{2.51}{Re \cdot \sqrt{\lambda}} \right)
\]

(3.22)

with \( \lambda \) - friction coefficient

\( Re \) - Reynolds number

The parabolic velocity distribution in the viscous sublayer and the logarithmic velocity distribution in the turbulent boundary layer are fully developed (see fig. 3.13). The shear stress distributions are linear.

![Fig. 3.13: Velocity and shear stress distribution](image)

3.4.2.2.2 Transitional behavior

The degree of turbulence of the flow increases with the Reynolds number. The fluid particles travel closer to the boundary in their movement transverse to the flow direction. The thickness \( \delta \) of the viscous sublayer becomes smaller with increasing Reynolds number (see formula (2.14)). If the sublayer is too thin for all roughness elements to be covered, the wall
roughness has an influence on the flow resistance and it has to be considered in the determination of the friction coefficient $\lambda$. 

$$\frac{1}{\sqrt{\lambda}} = -2.03 \cdot \log \left( \frac{2.51}{\text{Re} \cdot \sqrt{\lambda}} + \frac{k_s}{14.84 \cdot r_{hy}} \right)$$  \hspace{1cm} (3.23)$$

with $\lambda$ – friction coefficient

$\text{Re}$ – Reynolds number

$r_{hy}$ – hydraulic radius

$k_s$ – equivalent sand roughness

3.4.2.2.3 Hydraulically rough

With an even larger Reynolds number the thickness of the viscous sublayer shrinks until it has no effect anymore. Then the tenacity has no influence on the flow resistance anymore. The magnitude of the relative roughness alone has to be considered determining the friction coefficient $\lambda$. 

$$\frac{1}{\sqrt{\lambda}} = -2.03 \cdot \log \left( \frac{k_s}{14.84 \cdot r_{hy}} \right)$$  \hspace{1cm} (3.24)$$

with $\lambda$ – friction coefficient

$r_{hy}$ – hydraulic radius

$k_s$ – equivalent sand roughness

The velocity distribution in the turbulent boundary layer is logarithmic and the shear stress distribution is linear (see Fig. 3.14). The viscous sublayer is without effect.

Fig. 3.14: Velocity and shear stress distribution ➔ hydraulically rough
3.4.2.2.4 hydraulically extremely rough

The statements for hydraulically rough walls made in chapter 3.4.2.2.3 are valid only if the size of the roughness elements \( k \) is much smaller than the flow depth.

For a roughness that does not fulfill this requirement, which is called extremely rough, the flow resistance is caused primarily by form drag due to flow past the roughness elements. This depends both on the Reynolds number \( Re \) and on the Froude number \( Fr \). A two-layered flow occurs (see Fig. 3.15). The lower layer has an almost constant flow velocity. The second layer has a logarithmic velocity distribution.

Fig. 3.15: Flow conditions over extremely rough boundary

In extreme situations, in which roughness elements reach up to the water surface, the second layer is almost missing. Then the vertical velocity profile is totally distorted over the entire flow depth and the logarithmic resistance law loses its validity.

In extremely rough waters the value of the roughness coefficient \( k \) is not sufficient for the description of the flow resistance. Moreover, form and arrangement of the roughness elements in flow direction as well as perpendicular the flow direction play a major role in flow resistance. A hypothesis that accounts for these physical conditions has been developed by Aguirre-Pe/Fuentes (1990):

\[
\frac{1}{\sqrt{\lambda}} = 0.88 \frac{\beta_w \cdot d_m}{r_{hy}} + 2.03 \cdot \log \frac{11,1 \cdot r_{hy,j}}{\alpha_t \cdot d_m}
\]

(3.25)

with

- \( \lambda \) – friction coefficient
- \( r_{hy} \) – hydraulic radius
- \( d_m \) – mean diameter of the roughness elements
- \( \alpha_t \) – texture parameter which considers shape and arrangement of the roughness elements
- \( \beta_w \) - wake-parameter
This relationship considers the two aspects of the two-layered flow and consists of two terms. The first term describes the layer close to the bed, the second considers for the zone near the surface.

If the roughness element type is for example small vegetation (chapter 3.4.3.1) possibly nothing can be said about the flow conditions in the lower layer. The time-average of measured velocities also does not give a useful result. Unordered eddies and backflows characterize the lower layer. The largest shear stress $\tau_0$ arises in the transition zone, where the velocity gradient is maximal.

Fig. 3.16: Velocity and shear stress distribution ➔ hydraulically extremely rough

3.4.2.3 Consideration of additional influences in semi-natural flowing water

The equivalent sand roughness $k_s$ as used in pipe hydraulics or artificial channel calculations hardly applies to the discharge calculation of semi-natural flowing waters since the flow is subject to additional influences here. Additional influences are e.g.

- Changes in Cross-section,
- Sediment transport and
- Meandering.

These are not considered in the general discharge formula according to Darcy-Weisbach. They can be estimated over a head loss parameter $h_v$ or over a modified $k_s$-value.

In the second case, modifications to the equivalent sand roughness (the so-called basic roughness) $k_s$ have to be made. Only in case of small additional influences calculations can be done with basic roughnesses. Common basic roughnesses are specified in table 3.1. There is no known procedure for determining the magnitude of additional parameters. They are obtained through the evaluation of measurements in nature. For flowing waters with a similar character, similar parameters can be used.
3.4.3 $\lambda$ for roughness by vegetation

For hydraulic quantification the vegetation on banks and floodplains has to be classified. There is a differentiation between small, middle and high vegetation (Fig 3.17). The vegetation height of small vegetation is substantially smaller than the flow depth, so that the velocity distribution turns out according to a boundary layer flow. The flow is blocked by vegetation over the entire flow depth for middle and high vegetation, so that the collision loss increases with the flow depth. There is no logarithmic velocity distribution under these conditions, instead the flow velocity is constant over the flow depth. For this reason small, middle and high vegetation must be described with different approaches. According to their hydraulic effect a difference is made of flooded (submerged), in-flow (immersed) and isolated vegetation.

Fig. 3.17: Classification of vegetation

3.4.3.1 Friction coefficient $\lambda_K$ for flooded vegetation (small vegetation)

Field and laboratory investigations of Kouwen (1988) led to the listing of a resistance law for flooded vegetation, which puts the friction coefficient $\lambda_K$ in relation to vegetation height $k_G$, flow depth $h$ and the vegetation stiffness MEI of the vegetation elements (Fig. 3.18).

$$\frac{1}{\sqrt{\lambda_K}} = a + b \cdot \log(h_G/k_G)$$  \hspace{1cm} (3.26)

with $a, b$ – parameter from table 3.3

$h_G$ – vegetation height in starting position

$k_G$ – vegetation height under flow (Fig. 3.18)
\[ k_G = 0,14 \cdot h_G \cdot \left( \frac{MEI}{\tau_{So}} \right)^{0.25} \left( \frac{MEI}{h_G} \right)^{1.59} \]  \hspace{1cm} (3.27)

with \( MEI \) - vegetation stiffness

\[ MEI = 319 \cdot h_G^{3.3} \text{ for grass} \]  \hspace{1cm} (3.28)

\[ MEI = 25.4 \cdot h_G^{2.26} \text{ for dead grass} \]  \hspace{1cm} (3.29)

\( \tau_{So} \) - shear stress at the bottom

\[ \tau_{So} = \rho \cdot g \cdot \frac{A}{I_u} \cdot l_E \]  \hspace{1cm} (3.30)

Fig. 3.18: Flow conditions for flooded vegetation

The parameters \( a \) and \( b \) vary with the bending of the vegetation, which is expressed by the relationship of the critical shear stress velocity \( v_{krit}^* \) to the actual shear stress velocity \( v^* \). With the relationships

\[ v^* = \sqrt{g \cdot h \cdot l_E} \]  \hspace{1cm} (3.31)

and

\[ v_{krit}^* = \text{Minimum von } \left\{ \begin{array}{ll} 0.028 + 6.33 \cdot MEI \\
0.23 \cdot MEI^{0.106} \end{array} \right. \]  \hspace{1cm} (3.32)

the parameters \( a \) and \( b \) can be determined in table 3.3.
### Tab 3.3: Parameters a and b

In opposition to the vegetation height $h_G$ the parameter MEI is not directly determinable in nature. Kouwen (1990) therefore states formulas (3.28) and (3.29) for grass, so the parameter MEI can be derived approximately from the grass height $h_G$. For other flooded vegetation types like shrubs there are no empirical values for MEI. An exact but complex procedure for the direct determination of MEI in the field can be read about in literature by Kouwen (1990).

#### 3.4.3.2 Friction coefficient $\lambda_p$ for immersed vegetation (middle and high vegetation)

For the computation of the flow resistance $\lambda_p$ due to flow over vegetation the form drag of the vegetation elements is projected on the base area $(a_x \cdot a_y)$ of the vegetation elements (Fig. 3.19):

$$\lambda_p = \frac{4 \cdot h_p \cdot d_p \cdot c_{WR} \cdot \cos(\alpha_{lat})}{a_x \cdot a_y}$$

(3.33)

with $a_x$, $a_y$ – distance of vegetation elements (Fig. 3.19)

$h_p$ – vegetation height underwater (Fig. 3.17)

$d_p$ – vegetation diameter (Fig. 3.17)

$\alpha_{lat}$ – bank slope

$c_{WR}$ – form friction coefficient cylinder in group*

with $c_{WR} = 1.31 \cdot c_{Wc} \cdot \left( \frac{v_n}{v_{Vor}} \right)^2 + \Delta c_w$

(3.34)

with $c_{Wc}$ – form friction coefficient cylinder*

$v_n$ – flow velocity in front of vegetation*
\( v_{Vor} \) – velocity in the vegetation area*

\( \Delta c_W \) – wave-current form drag

* Explanations concerning the individual properties in the following text

Fig. 3.19: Immersed vegetation (middle and high vegetation)
The form friction coefficient $c_{WR}$ depends on the relationship of the initial flow velocity $v_n$ to the mean flow velocity in the vegetation area $v_{Vor}$, as well as the magnitude of the waves formed by the flow past the vegetation elements (Lindner, 1982).

After Pasche (1984) the relative initial velocity $v_n/v_{Vor}$ can be measured with the help of the following empirically derived equation:

$$\left(\frac{v_n}{v_{Vor}}\right)^2 = 1.15 \left(\frac{a_{NL}}{a_x}\right)^{-0.48} + 0.5 \left(\frac{a_{NB}}{a_y}\right)^{1.1}$$

(3.35)

with $a_{NL}$ – wake length

$$a_{NL} = 128.9 \cdot c_{w^\infty} \cdot d_p \cdot \left[1 + \frac{g \cdot a_{NL} \cdot l_e}{v_{Vor}^2/2}\right]^{-2.14}$$

(3.36)

$a_{NB}$ – wake width

$$a_{NB} = 0.24 \cdot a_{NL}^{0.59} \cdot (c_{w^\infty} \cdot d_p)^{0.41}$$

(3.37)
The form drag $c_{W\infty}$ corresponds to the form drag of an individual cylindrical vegetation element in an ideal, two-dimensional flow. Based on Wieselsberger’s diagram (Fig. 2.20) it amounts to:

$$c_{W\infty} = 3.07 \cdot \text{Re}_p^{-0.168} \quad \text{for} \quad \text{Re}_p = \frac{V_{vor} \cdot d_p}{v} < 800$$

$$c_{W\infty} = 1.0 \quad \text{for} \quad 800 < \text{Re}_p < 8000$$

$$c_{W\infty} = 1.2 \quad \text{for} \quad 8000 < \text{Re}_p < 10^5$$

The vegetation quantities $a_x$, $a_y$, and $d_p$ are geometrical (naturally determinable). The microstructure of the vegetation is the foundation (e.g. mean branch diameter and branch distance in a height of approx. 1 m above the willow head) for closed vegetation rows (e.g. willow rows). For individual shrubs and trees the macrostructure of the vegetation has to be taken into consideration (mean shrub/tree diameter and shrub/tree distance). These quantities can be determined either directly in nature according to DVWK (1991) or, if the effort is not justifiable, be approximated with help of empirical values as proposed in table 3.4. It has to be noted that the parameters can represent averaged values over the flow depth and thus vary for different discharge events. This is to be considered particularly for shrubs and dammed up wood.
<table>
<thead>
<tr>
<th>Vegetation</th>
<th>Level of growth</th>
<th>$d_p$ [m]</th>
<th>$a_x$ [m]</th>
<th>$a_y$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cane brake</td>
<td>–</td>
<td>0,007</td>
<td>0,02</td>
<td>0,02</td>
</tr>
<tr>
<td>Bushes</td>
<td>one year</td>
<td>0,03</td>
<td>0,3</td>
<td>0,3</td>
</tr>
<tr>
<td></td>
<td>more years</td>
<td>0,045</td>
<td>0,2</td>
<td>0,2</td>
</tr>
<tr>
<td>Reed</td>
<td>dense</td>
<td>0,01</td>
<td>0,05</td>
<td>0,05</td>
</tr>
<tr>
<td>Bushes, even</td>
<td>small</td>
<td>0,3</td>
<td>1,5</td>
<td>1,5</td>
</tr>
<tr>
<td>distribution</td>
<td>middle</td>
<td>0,4</td>
<td>3,0</td>
<td>3,0</td>
</tr>
<tr>
<td></td>
<td>high</td>
<td>0,8</td>
<td>3,0</td>
<td>3,0</td>
</tr>
<tr>
<td></td>
<td>spaced</td>
<td>0,1</td>
<td>4,0</td>
<td>4,0</td>
</tr>
<tr>
<td></td>
<td>in small groups</td>
<td>0,5</td>
<td>6,0</td>
<td>6,0</td>
</tr>
<tr>
<td></td>
<td>in large groups</td>
<td>1,0</td>
<td>7,0</td>
<td>7,0</td>
</tr>
<tr>
<td></td>
<td>with treetop</td>
<td>0,03</td>
<td>0,15</td>
<td>0,15</td>
</tr>
<tr>
<td>Trees</td>
<td>spaced</td>
<td>0,5</td>
<td>10,0</td>
<td>10,0</td>
</tr>
<tr>
<td></td>
<td>dense, young</td>
<td>0,2</td>
<td>5,0</td>
<td>5,0</td>
</tr>
<tr>
<td></td>
<td>old</td>
<td>0,4</td>
<td>5,0</td>
<td>5,0</td>
</tr>
<tr>
<td></td>
<td>dense, old</td>
<td>0,8</td>
<td>5,0</td>
<td>5,0</td>
</tr>
</tbody>
</table>

Tab 3.4: Vegetation quantities for different vegetation types

3.4.4 $\lambda$ as a function of the cross-section

The hypothesis for the modeling of bed and boundary roughnesses introduced in chapters 3.4.1 and 3.4.2, as well as the parameters used, have been developed for a circular flow cross-section. An even distribution of the wall shear stresses over the wetted outline is assumed. Corrections have to be made if the flow cross-section deviates from the circular shape and the associated regular flow conditions.

Type and complexity of considerations about the open channel shape strongly depend on the question if the flow conditions are approximately the same over the entire flow cross-section. There is a differentiation between two cases:

2. If the flow velocity varies significantly over the cross-section due to the channel shape, the cross-section can be subdivided for hydraulic computation. The procedure for a partitioned cross-section is described in chapter 3.4.6.

3. If the same flow velocities arise in the entire flow cross-section and if they correspond approximately to the mean flow velocity $v = Q / A_{ges}$, this is called a compact cross-section. In this case the influence of the cross-sectional shape (deviating from a
circular shape) can be considered with help of a form coefficient \( f \) in the Colebrook-White resistance law (chapter 3.4.1)

\[
\frac{1}{\sqrt{\lambda}} = -2.03 \cdot \log \left( \frac{2.51}{f \cdot \text{Re} \sqrt{\lambda}} + \frac{k_s}{f \cdot 14.84 \cdot r_{hy}} \right)
\]

with \( \lambda \) – friction coefficient
\( f \) – form coefficient
\( \text{Re} \) – Reynolds number
\( r_{hy} \) – hydraulic radius
\( k_s \) – equivalent sand roughness

The form coefficient \( f \) can be estimated with the help of the following relations:

**Rectangular cross-sections:**
\[
f = 0.9 - 0.38 \cdot e^{-5h/b}
\]

(3.40)

**Triangular cross-sections with smooth boundary:**
\[
f = 1.276 \cdot \left( \frac{h}{b_{ap}} \right)^{3/20}
\]

(3.41)

**Trapezoidal cross-sections with smooth boundary:**
\[
f = 1.13 \cdot \left( \frac{r_{hy}/b_{so}}{1/4} \right)
\]

(3.42)

### 3.4.5 \( \lambda \) for layering of roughnesses

Generally there are very heterogeneous roughness conditions in flowing waters. Apart from a variation in flow direction the roughness structures can also be very different over the cross-section. Two cases must be differentiated in analogy to chapter 3.4.4:

- If the flow velocity varies significantly over the cross-section due to the channel shape, the cross-section can be subdivided for hydraulic computation. The procedure for a partitioned cross-section is described in chapter 3.4.6.

- If the flow velocity in all points of a cross-section can be approximated by the mean velocity of the cross-section then the different roughness zones can computationally be overlaid to the total roughness. One differentiates between the overlapping phenomena explained in the following two chapters:

#### 3.4.5.1 Sections with different roughness

The roughness varies in sections along the wetted perimeter (Fig 3.22). After Einstein (1934) the friction coefficients of the individual sections can be overlaid linearly to a total friction coefficient if \( v_i = \overline{v} \approx 1.0 \).
\[ \lambda_{\text{ges}} = \sum_{j=1}^{n} \frac{\lambda_j l_{u,j}}{l_{u,\text{ges}}} \]  

with \( j = 1,2...n \)

- \( n \) – number of roughness sections \( j \)
- \( \lambda \) – friction coefficient

\[ l_{u,j} = \frac{A_j}{r_{hy,j}} = \text{wetted perimeter for roughness section } j \]

\[ l_{u,\text{ges}} = \frac{A_{\text{ges}}}{r_{hy,\text{ges}}} = \text{total wetted perimeter} \]

Fig. 3.22: Cross-section with different roughnesses

Segments \( A_j \) are assigned to each of these sections. There is no shear stress at their interfaces (the separation lines are perpendicular to the isotachs). The size and the hydraulic radius of these segments are proportional to the friction coefficient of the section:

\[ \frac{\lambda_j}{\lambda_{\text{ges}}} = \frac{r_{hy,j}}{r_{hy,\text{ges}}} \]  

with \( \lambda \) – friction coefficient

\( r_{hy} \) – hydraulic radius

The friction coefficient \( \lambda_j \) is calculated using the hydraulic radius of the segments \( r_{hy,j} \):

\[ \frac{1}{\sqrt{\lambda_j}} = -2.03 \cdot \log \left( \frac{2.51}{\text{Re} \sqrt{\lambda_j}} + \frac{k_{s,j}}{14.84 \cdot r_{ny,j}} \right) \]  

with \( \lambda \) – friction coefficient
3.4.5.2 Overlapping of roughness structures

According to Lindner (1982), if different roughness structures appear together locally, e.g. submerged vegetation and immersed trees, the flow resistance caused by individual structures in smooth and rough conditions can be overlaid linearly:

\[
\lambda_{ges} = \sum_{j=1}^{n} \lambda_j
\]  

(3.46)

For extremely rough conditions the reliability of this method is not guaranteed. Since no flow cross-sections can be assigned to vertically overlaying roughnesses, the friction coefficient \(\lambda_j\) is calculated for the entire cross-section. In this case the hydraulic radius \(r_{ny,i}\) of the segment is not to be used in the Colebrook-White resistance law, but the hydraulic radius of the cross-section \(r_{ny,ges}\) in which the common roughness structures are located (Fig 3.23).

Fig. 3.23: Cross-section with vertically varying roughness
3.4.6 $\lambda$ for partitioned cross-sections

A partition of the flow cross-section is necessary whenever clear differences of local flow velocities in the cross-section arise due to the cross-sectional shape (see chapter 3.4.4) or substantial differences of bed roughness (see chapter 3.4.5). Generally such conditions are:

- River-floodplain-flow (see chapter 3.4.6.1) and
- Flowing waters with vegetation on bank or floodplain (see chapter 3.4.6.2)

An intensive momentum exchange between the individual cross-sections is the result (Fig. 3.24), which becomes apparent in a substantial decrease of the discharge. This momentum exchange (interaction) is the case since momentum flows of different velocity in the floodplain, vegetation zone and river cause considerable turbulent shear stresses at their interfaces because of the tenacity of the water. These intensive turbulences lead to a shift of parts of the momentum from the river toward floodplain and tree zone. When, as a reaction, a part of the slow floodplain momentum shifts to the river, macroscopic coherent eddy structures occur. These remain to exist for a relatively long time before they disintegrate into smaller turbulence structures, until they finally convert the mechanical energy of the turbulence into heat energy by internal friction.

![Visualized flow in partitioned channel (from Pasche, 1984)](image)

The flow resistance caused by momentum exchange can be considered by the fact that the interface between the cross-sections of different flow velocities is understood as an imaginary boundary influenced by shear stress relative to the intensity of the momentum exchange, and thus the interface has a flow resistance (Fig. 3.25).
3.4.6.1 Partition due to geometry (river-floodplain-flow)

The hypothesis of an imaginary vertical partition with a friction coefficient $\lambda_T$ became generally accepted for the computation of cross-sections. To be able to calculate the discharge for the entire cross-section a computation of the discharges of the segments using the Darcy-Weisbach flow law is necessary. With approximately the same bed roughnesses of the cross-sections segments the interface roughness $\lambda_T$ after Könemann is:

$$\lambda_T = \lambda_{So,Mainchannel}$$  \hspace{1cm} (3.47)

The roughness of the imaginary interface only has to be considered for the computation of the discharge for the main channel (Fig 3.26).

Fig. 3.25: Shear stress and velocity distribution in a partitioned channel

Fig. 3.26: Geometrically partitioned cross-section
3.4.6.2 Partition due to vegetation

The procedure after Könemann (chapter 3.4.6.1) cannot be used here since the bed roughnesses of the partitions are generally very different. If a partition of the cross-section due to vegetation is necessary, an imaginary interface between vegetation zone and vegetation-free zone with a friction coefficient $\lambda_T$ is assumed. The interface friction coefficient $\lambda_T$ can be computed with the following formula:

$$\frac{1}{\sqrt{\lambda_T}} = -2 \cdot \log \left[ 0.07 \cdot \left( \frac{c \cdot b_m}{b_{ll}} \right)^{1.07} \cdot \Omega \right]$$

with $\Omega$ - vegetation parameter:

$$\Omega = 0.07 \cdot \left( \frac{a_{NL}}{a_x} \right)^{3.3} + \left( \frac{a_{NB}}{a_y} \right)^{0.95}$$

with $a_x, a_y$ from Fig. 3.19

$$a_{NL} = 128.9 \cdot c_w \cdot d_p \cdot \left[ 1 + \frac{g \cdot a_{NL} \cdot l}{\sqrt{V_T^2/2}} \right]^{-2.14}$$

$$a_{NB} = 0.24 \cdot a_{NL}^{0.59} \cdot (c_w \cdot d_p)^{0.41}$$

$a_{NL}$ – wake length and $a_{NB}$ – wake width

(individual quantities see chapter 3.4.3.2)

c – cross-section coefficient:

c = 1.7 compact cross-sections with vegetation

c = 1.0 for partitioned cross-sections

$b_m$ – vegetation width (Fig. 3.27)

$$b_m = \frac{h_T}{\lambda_V \left( 0.068 e^{0.564 c_T} - 0.056 \right)}$$

with $h_T$ – interface height

$\lambda_V$ – floodplain friction coefficient

$\lambda_V = \lambda_w$ (after Kouwen) for small vegetation

$\lambda_V = \lambda_p + \lambda_{so,vor}$ with $\lambda_p$ for high vegetation

$c_T$ – dimensionless interface flow velocity
\[ c_T = -3.27 \cdot \log \Omega + 2.85 \]  \hspace{1cm} (3.55)

\( b_{\text{III}} \) – main channel width (Fig. 3.27)

\[ b_{\text{III,li}} = b_{\text{III,re}} = \frac{b_F}{2} \] - for symmetrical flow with

\[ \frac{b_F}{h_T} < 50 \]  \hspace{1cm} (3.56)

(for other requirements refer to the BWK)

**Fig. 3.27**: Influence zones in partitioned cross-section
4 Flow over hydraulic structures

The derivations in chapter 4.1.2 are taken from [Knapp], all other derivations are according to BWK, 1999.

4.1 Hydraulic loss at weirs

4.1.1 Introduction

The loss occurring at weirs can be calculated in dependence of the flow state and/or the cross-sectional shape.

If the loss is calculated in dependence of the flow state, the following three types of discharge are differentiated:

- free (critical) overflow (downstream water level significantly below crest)
- partially free (subcritical) overflow (downstream water level just below crest)
- submerged weir (downstream water level above crest, no visible hydraulic jump)

Fig. 4.1: free overflow
Fig. 4.2: partially free overflow

Abb. 4.3 submerged weir

If weirs are calculated in dependence of the cross-sectional shape, they are differentiated as:

- round-crested weirs
- sharp-crested weirs
- broad-crested weirs
Fig. 4.4 cross-sectional shapes: a) round-crested, b) sharp-crested, c) broad-crested

4.1.2 Derivation of overflow equations

The following derivations have been taken from [Knapp].

4.1.2.1 Derivation of formula for free overflow after Poleni-Weisbach

Fig. 4.5 Derivation of the Poleni-Weisbach equation
According to Toricelli:

\[ v_2 = \sqrt{2g \left( h + \frac{v_1^2}{2g} \right)} \]  \hfill (4.1)

The local nappe velocities are expected to change according to Toricelli’s law. The discharge \( dQ \) flows through an area \( b \cdot dh \). This leads to:

\[ dQ = v_2 \cdot b \cdot dh = b \sqrt{2g \left( h + \frac{v_1^2}{2g} \right)} dh \]  \hfill (4.2)

Integration yields:

\[ Q = \int_{0}^{h} dQ = b \sqrt{2g} \cdot \int_{0}^{h} \left( h + \frac{v_1^2}{2g} \right)^{1/2} dh \]  \hfill (4.3)

\[ Q = b \sqrt{2g} \cdot \left. \frac{1}{1.5} \left( h + \frac{v_1^2}{2g} \right)^{1.5} \right|_{0}^{h_0} \]  \hfill (4.4)

\[ Q = \frac{2}{3} b \sqrt{2g} \cdot \left( h_0 + \frac{v_1^2}{2g} \right)^{3/2} - \left( \frac{v_1^2}{2g} \right)^{3/2} \]  \hfill (4.5)

The influences not taken into consideration in this formula are considered with the coefficient \( \mu_w \).

\[ Q = \mu_w \frac{2}{3} b \sqrt{2g} \cdot \left( h_0 + \frac{v_1^2}{2g} \right)^{3/2} - \left( \frac{v_1^2}{2g} \right)^{3/2} \]  \hfill (4.6)

If the upstream velocity \( v_1 \) is neglected, the formula is:

\[ Q = \mu_w \frac{2}{3} b \sqrt{2g} \cdot h_0^{3/2} \]  \hfill (4.7)
Critical reflection on the Poleni-Weisbach equation

Assumptions made in the derivation of the Poleni-Weisbach equation are mutually exclusive. For integration between the limits zero and \( h_0 \) it is necessary that the locally variable velocities \( v_2 \) are horizontal and that the nappe has a head of \( h_0 \) above the crest. However, parallel flow sections above the crest cause a hydrostatic pressure distribution that leads to a constant velocity \( v_2 \) in the nappe cross-section above the crest, which does not conform to Toricelli’s equation.

Moreover, the derivation requires a flow without loss. Even if the loss caused by the inlet and by friction is relatively small, it cannot be neglected for a precise solution of the problem.

Fig. 4.6 Overflow discharge diagram according to assumptions of the Poleni-Weisbach equation
4.1.2.2 Derivation of the general free overflow equation with consideration of the nappe curvature

Fig. 4.7 Pressure and velocity states at top of round-crested overflow

According to Toricelli:

\[ v = \sqrt{2g \cdot h_E} \]  \hspace{1cm} (4.8)

Therefore the velocity at the surface of the nappe is:

\[ v_w = \sqrt{2g \cdot (h_{E,w} - h_w)} \]  \hspace{1cm} (4.9)

At the surface of the crest, considering a remaining pressure \( \frac{p_k}{\rho g} \), the velocity is as follows:

\[ v_k = \sqrt{2g \cdot \left(h_{E,w} - \frac{p_k}{\rho g}\right)} \]  \hspace{1cm} (4.10)

This velocity is less than Toricelli’s velocity and is only equal if the remaining-pressure \( \frac{p_k}{\rho g} \) disappears.
This is the case for an aerated sharp-crested weir, for which the air pressure is at the bottom side of the nappe. The same conditions appear at a spillway with a crest similar to the nappe shape of an aerated sharp-crested weir.

The quotient $\Phi$ of the velocities at the top and bottom side of the nappe is:

$$\Phi = \frac{v_w}{v_k} = \sqrt{\frac{h_{E,w} - h_w}{h_{E,w} - \frac{p_k}{\rho g}}}$$

(4.11)

This formula in a different form:

$$\Phi^2 h_w - \Phi^2 \frac{p_k}{\rho g} = h_{E,w} - h_w$$

(4.12)

Furthermore, if a pressure coefficient $\beta$ is introduced, which describes the relationship of the pressure and the nappe depth at the crest, the following equation is obtained with $\beta = \frac{p_k}{\rho g h_w}$:

$$\Phi^2 h_w - \Phi^2 \beta \cdot h_w = h_{E,w} - h_w$$

(4.13)

This formula can be solved for the nappe depth $h_w$:

$$h_w = \frac{h_{E,w} (1 - \Phi^2)}{(1 - \Phi^2 \beta)}$$

(4.14)

The principle of angular momentum is now applied to the overflow problem thus considering the streamline curvature. According to the principle of angular momentum the product of flow velocity $v$ and distance from any center of curvature $r$ is constant.

In this case the following formula applies for any velocity in distance $r$:

$$v = \frac{v_k \cdot r_k}{r}$$

(4.15)

The velocity ratio $\Phi$ therefore is:

$$\Phi = \frac{v_w}{v_k} = \frac{v_k \cdot r_k}{r_w} \frac{r_k}{r_w} \frac{r_k}{r_w + h_w}$$

(4.16)

Solving for the distance $r_k$ yields:
\[ r_k = \frac{\Phi \cdot h_w}{1 - \Phi} \]  

(4.17)

Substituting the nappe depth into this formula and solving for \( r_k \) gives the following formula:

\[ r_k = \frac{\Phi}{1 - \Phi} \cdot \frac{h_{E,w}(1 - \Phi^2)}{(1 - \Phi^2 \beta)} = h_{E,w} \cdot \frac{\Phi(1 + \Phi)}{(1 - \Phi)(1 - \Phi^2 \beta)} = h_{E,w} \cdot \frac{\Phi(1 + \Phi)}{(1 - \Phi^2 \beta)} \]  

(4.18)

Together with the equations stated below

\[ v = \frac{v_k \cdot r_k}{r} \]  

(4.19)

\[ v_k = \sqrt{2g \cdot \left(h_{E,w} - \frac{p_k}{\rho g}\right)} \]  

(4.20)

the local velocity \( v \) in the nappe is:

\[ v = \frac{1}{r} \sqrt{2g \cdot \left(h_{E,w} - \frac{p_k}{\rho g}\right) \cdot h_{E,w} \cdot \Phi(1 + \Phi)} \]  

(4.21)

The two equations

\[ \beta = \frac{p_k}{\rho g} \cdot \frac{1}{h_w} \quad \text{and} \quad h_w = \frac{h_{E,w}(1 - \Phi^2)}{(1 - \Phi^2 \beta)} \]  

(4.22)

help finding a formula for the remaining-pressure:

\[ \frac{p_k}{\rho g} = \beta \cdot h_{E,w} \frac{(1 - \Phi^2)}{(1 - \Phi^2 \beta)} \]  

(4.23)

The expression for the local nappe velocity can now be put into a different form:

\[ v = \frac{1}{r} \sqrt{2gh_{E,w} \cdot \left(1 - \beta \cdot \frac{(1 - \Phi^2)}{(1 - \Phi^2 \beta)}\right) \cdot h_{E,w} \cdot \Phi(1 + \Phi)} \]  

(4.24)

\[ v = \frac{1}{r} \sqrt{2g} \cdot \left(h_{E,w}\right)^{\frac{3}{2}} \cdot \frac{\Phi(1 + \Phi)}{(1 - \Phi^2 \beta)} \cdot \left(\frac{1 - \Phi^2 \beta}{(1 - \Phi^2 \beta)}\right)^\frac{1}{2} \]  

(4.25)

\[ v = \frac{1}{r} \sqrt{2g} \cdot \left(h_{E,w}\right)^{\frac{3}{2}} \cdot \frac{\Phi(1 + \Phi)(1 - \beta)^{\frac{1}{2}}}{(1 - \Phi^2 \beta)^{\frac{3}{2}}} \]  

(4.26)
The total discharge is now obtained by integration:

\[ Q = \int_{r_k}^{r_e} (b \cdot V) \, dr = b \cdot \sqrt{2g} \cdot (h_{E,w})^{3/2} \cdot \frac{\Phi (1+\Phi)(1-\beta)^{1/2}}{(1-\Phi^2\beta)^{3/2}} \int_{r_k}^{r_e} 1 \, dr \]  

(4.27)

With the help of the principle of angular momentum in a different form

\[ \ln \left( \frac{r_w}{r_k} \right) = \ln \left( \frac{v_x}{v_w} \right) = \ln \left( \frac{1}{\Phi} \right) \]  

(4.28)

the total discharge is calculated as:

\[ Q = b \cdot \sqrt{2g} \cdot (h_{E,w})^{3/2} \cdot \frac{\Phi (1+\Phi)(1-\beta)^{1/2}}{(1-\Phi^2\beta)^{3/2}} \ln \left( \frac{r_w}{r_k} \right) \]

\[ = b \cdot \sqrt{2g} \cdot (h_{E,w})^{3/2} \cdot \frac{\Phi (1+\Phi)(1-\beta)^{1/2}}{(1-\Phi^2\beta)^{3/2}} \ln \left( \frac{1}{\Phi} \right) \]

(4.29)

\[ = b \cdot (h_{E,w})^{3/2} \cdot C \]

The result is that the discharge depends on the effective energy head \( H_w \), the ratio \( \Phi \) of the side velocities in the nappe, and the pressure coefficient \( \beta \).

Fig. 4.8 Nappe velocity direction and discharge area
Note:

The application of the principle of angular momentum is not exact since the nappe velocities are assumed to be horizontal. The velocity direction at the nappe surface is not horizontal, however, the error is marginal.

The formula for the discharge is now solved for the energy head:

\[
h_{E,w} = \left[ \frac{Q}{b \cdot \sqrt{2g \cdot (1 - \beta)^2}} \right]^{2/3} \left( 1 - \Phi^2 \beta \right) \left[ \Phi (1 + \Phi) \right]^{2/3} \left[ \ln \left( \frac{1}{\Phi} \right) \right]^{2/3} \tag{4.30}
\]

For any given pressure coefficient \( \beta \) the discharge assumes a value in a way that the effective energy head and thus the amount of overflow water has a maximum, i.e.:

\[
\frac{\partial h_{E,w}}{\partial \Phi} = 0 \tag{4.31}
\]

This leads to:

\[
\frac{\partial h_{E,w}}{\partial \Phi} = \frac{\partial}{\partial \Phi} \left[ \left( \frac{Q}{b \cdot \sqrt{2g \cdot (1 - \beta)^2}} \right)^{2/3} \left( 1 - \Phi^2 \beta \right) \left[ \Phi (1 + \Phi) \right]^{2/3} \left[ \ln \left( \frac{1}{\Phi} \right) \right]^{2/3} \right] \tag{4.32}
\]

Further transformations of the formula yield the following expression:

\[
\frac{\partial h_{E,w}}{\partial \Phi} = \frac{Q}{b \cdot \sqrt{2g \cdot (1 - \beta)^2}} \left[ \frac{2}{3} \left( \ln \left( \frac{1}{\Phi} \right) \right)^{2/3} \left( 1 - \Phi^2 \beta \right) \left[ \Phi (1 + \Phi) \right]^{2/3} \left( 1 + 2 \Phi + \beta \Phi^2 (\Phi + 2) \right) \right] = 0 \tag{4.33}
\]

This equation can be solved for the pressure coefficient \( \beta \):

\[
\beta = \frac{1}{\Phi^2 \left( 1 + \Phi + (2 + \Phi) \cdot \ln \left( \frac{1}{\Phi} \right) \right)} \tag{4.34}
\]

If the discharge over the weir has a maximum, there is a well-defined relationship between the pressure coefficient \( \beta \) and the velocity ratio \( \Phi \)!
Discussion of the pressure coefficient $\beta$

1. Case:

For parallel stream lines, e.g. in the case of a broad-crested weir, the pressure coefficient is $\beta = 1$, i.e. the pressure distribution in the nappe is hydrostatic. This means that the velocity ratio $\Phi$ is also 1, so all stream velocities are equal.

With the relationship

$$r_k = h_{E,w} \frac{\Phi (1 + \Phi)}{(1 - \Phi^2 \beta)}$$  \hspace{1cm} (4.35)

the ratio of the weir crest radius to the energy head can be determined:

$$\frac{r_k}{h_{E,w}} = \frac{\Phi (1 + \Phi)}{(1 - \Phi^2 \beta)} = \frac{2}{0} = \infty$$  \hspace{1cm} (4.36)

This relationship shows that for plane stream lines the weir crest has to be plane.

Furthermore, a relationship between the nappe depth and the energy head can be found:

$$\frac{h_w}{h_{E,w}} = \frac{(1 - \Phi^2)}{(1 - \Phi^2 \beta)} = 0 \rightarrow \frac{h_w}{h_{E,w}} = \frac{2}{3}$$  \hspace{1cm} (4.37)

The limit of this quotient is 2/3, corresponding to the critical depth. Now the assumption that the discharge over the weir takes place requiring the least energy is confirmed.

Moreover the discharge coefficient $C$ can be determined:

$$C = \sqrt{2g \cdot \frac{\Phi (1 + \Phi) (1 - \beta)^{1/3}}{(1 - \Phi^2 \beta)^{2/3}} \cdot \ln \left( \frac{1}{\Phi} \right)} = 1.7048$$  \hspace{1cm} (4.38)

2. Case:

The pressure coefficient for an open sharp-crested weir or a correspondingly shaped spillway is $\beta = 0$, since the pressure below the nappe is equal to the air pressure. The velocity ratio is obtained as $\Phi = 0.4685$ with the help of an iterative approach.

Thus the ratio of the weir crest radius to the energy head is:

$$\frac{r_k}{h_{E,w}} = \frac{\Phi (1 + \Phi)}{(1 - \Phi^2 \beta)} = 0.4685 (1 + 0.4685) = 0.6879$$  \hspace{1cm} (4.39)
Furthermore the ratio of the nappe depth to the energy head is:

\[
\frac{h_{w}}{h_{E,w}} = \frac{(1 - 0.4685^2)}{(1 - 0.4685^2 \beta)} = 0.7805
\]  

(4.40)

The discharge coefficient C is calculated as:

\[
C = \sqrt{2g} \cdot \Phi \frac{(1 + \Phi)(1 - \beta)^{\frac{1}{3}} \cdot \ln \left( \frac{1}{\Phi} \right)}{(1 - \Phi^2 \beta)^{\frac{1}{3}}} = 2.3107
\]  

(4.41)

4.1.2.3 Derivation of the formula for the subcritical overflow according to DuBuat

Fig. 4.9 Assumption about the velocity distribution at the weir crest according to DuBuat

According to DuBuat the discharge is divided into two parts, the former representing the free overflow between upstream and downstream water level and the latter calculated as a discharge through an orifice. An area of \( b \cdot h_u \) and a pressure head of \( H_u + \frac{v_u^2}{2g} - h_u \) is
assumed for the latter. According to the assumption the pressure distribution at the crest can be seen in Fig. 9.

The application of the Poleni-Weisbach equation yields the discharge formula for the free flow:

$$Q_1 = \mu_{W,1} \cdot \frac{2}{3} \cdot b \cdot \sqrt{2g} \left[ (h_{e,o} - h_u)^{\frac{3}{2}} - \left( \frac{v_a^2}{2g} \right)^{\frac{3}{2}} \right]$$  \hspace{1cm} (4.42)

The discharge through the orifice is calculated as:

$$Q_2 = \mu_{W,2} \cdot b \cdot h_u \cdot \sqrt{2g(h_{e,o} - h_u)}$$  \hspace{1cm} (4.43)

Thus the total discharge is:

$$Q = \mu_{W,1} \cdot \frac{2}{3} \cdot b \cdot \sqrt{2g} \left[ (h_{e,o} - h_u)^{\frac{3}{2}} - \left( \frac{v_a^2}{2g} \right)^{\frac{3}{2}} \right] + \mu_{W,2} \cdot b \cdot h_u \cdot \sqrt{2g(h_{e,o} - h_u)}$$  \hspace{1cm} (4.44)

In practice, it is not possible to get exact values for both coefficients. Rather a mean value for both coefficients is assumed and the velocity head $\left( \frac{v_a^2}{2g} \right)$ is neglected. This leads to:

$$Q = \mu \cdot b \cdot \sqrt{2g} \left[ \frac{2}{3} (h_{e,o} - h_u)^{\frac{3}{2}} + h_u \cdot (h_{e,o} - h_u)^{\frac{1}{2}} \right]$$  \hspace{1cm} (4.45)

$$Q = \mu \cdot b \cdot \sqrt{2g} \cdot (h_{e,o} - h_u)^{\frac{1}{2}} \left[ \frac{2}{3} h_{e,o} + \frac{1}{3} h_u \right]$$  \hspace{1cm} (4.46)

**Reflections about DuBuat’s overflow formula**

In the derivation of DuBuat’s formula the velocity distribution for the discharge area above the downstream water level is not correct according to Toricelli’s law, and a constant velocity for the area below the water level has never been proven by measurements. The derivation of the formula makes assumptions that do not correspond to real conditions.

Furthermore the water level above the crest decreases for subcritical overflow as well, i.e. the assumed discharge area is too large and the yielded discharge would be too large if the coefficients are not corrected.
4.1.2.4 Derivation of the general formula for subcritical overflow considering the stream line curvature

Fig. 4.10 General formula for the subcritical overflow

If the downstream water level rises well above the crest, the velocity distribution in the nappe becomes more even, up to the point of submersion where the flow over the crest is almost uniform.

In place of the velocities $w$ and $k$ a mean velocity $v_m$ is introduced. The pressure line corresponding to this velocity is located by $m \cdot h_w$ above the crest with $m$ being an unknown coefficient for now, that depends on the downstream water level. The energy equation for the subcritical overflow is therefore:

$$h_{E,w} = m \cdot h_w + \frac{v_m^2}{2g} = m \cdot h_w + \frac{1}{2g} \left( \frac{Q}{b \cdot h_w} \right)^2$$  \hspace{1cm} (4.47)

For any given discharge $Q$ and a certain nappe depth $h_w$ there is a minimum of the energy head, i.e. \( \frac{\partial H_w}{\partial h_w} = 0 \).

Thus its derivative is:
\[ \frac{\partial h_{E,w}}{\partial h_w} = m + \frac{1}{2g} \frac{Q^2}{b^2} h_w^3 = m - \frac{Q^2}{b^2} \frac{1}{g \cdot h_w^3} = 0 \]  

(4.48)

Transformation solving for the nappe depth yield:

\[ h_w = \sqrt[3]{\frac{Q^2}{g \cdot b^2 - m}} = \frac{h_{gr}}{\sqrt[3]{m}} \quad \text{mit} \quad h_{gr} = \sqrt[3]{\frac{Q^2}{g \cdot b^2}} \]  

(4.49)

This relationship can be used in the energy equation:

\[ h_{E,w} = m \cdot \frac{h_{gr}}{\sqrt[3]{m}} + \frac{1}{2g} \left( \frac{Q}{b} \right)^2 \frac{m^2}{h_{gr}^3} = m \cdot \frac{h_{gr}}{\sqrt[3]{m}} + \frac{1}{2} h_{gr} \frac{\sqrt[3]{m^2}}{h_{gr}^2} \]  

(4.50)

Further transformations lead to:

\[ h_{E,w} = h_{gr} \left( \frac{m}{\sqrt[3]{m}} + \frac{1}{2} \frac{\sqrt[3]{m^2}}{m^2} \right) = h_{gr} \left( 1.5 \cdot \frac{m}{\sqrt[3]{m}} \right) = 1.5 \cdot m^\frac{2}{3} \cdot h_{gr} \]  

(4.51)

The dimensionless energy head \( \tau_0 \) is introduced with:

\[ \tau_0 = \frac{h_{E,w}}{h_{gr}} = 1.5 \cdot m^{\frac{2}{3}} \]  

(4.52)

It only depends on the coefficient \( m \).

Thus, for any given weir shape and amount of discharge, the coefficient \( \tau_0 \) only depends on the downstream head \( h_u \) and therefore also on the dimensionless downstream head \( \tau_u = \frac{h_u}{h_{gr}} \). The general discharge formula for the subcritical overflow expressed with dimensionless properties is:

\[ \tau_0 = f(\tau_u) \]  

(4.53)

The functional relationship between these two properties will be examined for the two cases of free overflow and submerged flow in the following:
1. case: Equation for the free overflow with dimensionless properties

The equation for the free overflow is:

\[ Q = b \cdot h_{E,w}^{\frac{3}{2}} \cdot C \]  

(4.54)

Furthermore the equation for the critical depth can be solved for the discharge \( Q \):

\[ h_{gr} = \frac{Q^2}{\sqrt{g \cdot b^2}} \quad \rightarrow \quad Q = b \sqrt{g} \cdot (h_{gr})^{\frac{3}{2}} \]  

(4.55)

The two formulas are set equal and transformed:

\[ b \cdot (h_{E,w})^{\frac{3}{2}} \cdot C = b \sqrt{g} \cdot (h_{gr})^{\frac{3}{2}} \]  

(4.56)

\[ \left( \frac{h_{E,w}}{h_{gr}} \right)^{\frac{3}{2}} = \frac{\sqrt{g}}{C} \]  

(4.57)

The dimensionless energy head is defined as \( (\tau_0)_0 \):

\[ (\tau_0)_0 = \frac{h_{E,w}}{h_{gr}} = \frac{3\sqrt{g}}{C^2} = \frac{2,141}{2} \]  

(4.58)

2. case: Equation for the submerged flow with dimensionless properties

![Submerged flow diagram](image_url)

Fig. 4.11 Submerged flow
Neglecting losses at the inlet and assuming a mean velocity at the crest \( v_m \) the energy equation is:

\[
h_{E,\bar{h}} = h_w + \frac{v_w}{2g} = h_w + \frac{1}{2g} \left( \frac{Q}{b \cdot h_w} \right)^2
\]  
(4.59)

Using the critical depth \( h_{gr} \) for parallel stream lines in the same direction:

\[
(h_{gr})^3 = \frac{Q^2}{g \cdot b^2}
\]  
(4.60)

the energy equation can be written as follows:

\[
h_{E,\bar{h}} = h_w + \frac{1}{2} \frac{h_{gr}^3}{h_w^3}
\]  
(4.61)

If furthermore the drop of the water level at the crest is assumed to be very small, the nappe depth can be set equal to the downstream water level:

\[
h_{E,\bar{h}} = h_u + \frac{1}{2} \frac{h_{gr}^3}{h_u^3}
\]  
(4.62)

Dividing by \( h_{gr} \) and transformation yields the equation for the submerged flow with dimensionless properties:

\[
\frac{h_{E,\bar{h}}}{h_{gr}} = \frac{h_u}{h_{gr}} + \frac{1}{2} \frac{h_{gr}^3}{h_u^3} \quad \rightarrow \quad \tau_o = \tau_o + \frac{1}{2} \frac{1}{\tau_u^2}
\]  
(4.63)

The following linear relationship has been obtained from experiments and marks the boundary between free and subcritical overflow:

\[
\tau_o = 1,725 - 0,525 \cdot \tau_u
\]  
(4.64)

The derived relationships with dimensionless energy heads can be represented in a \( \tau_o - \tau_u \) diagram. This helps differentiating between free and subcritical overflow as well as submerged flow.

The equation for the free overflow is:

\[
(\tau_o)_0 = \frac{2,141}{2 \cdot C^3}
\]  
(4.65)
and it is shown as a horizontal in the diagram. This equation is valid up to the boundary to the subcritical overflow determined by

$$\tau_0 = 1,725 - 0,525 \cdot \tau_u \quad (4.66)$$

The transformation of this equation yields the limit:

$$\frac{\tau_u}{0,525} = 3,286 - 1,905 \frac{\tau_0}{0,525} \quad (4.67)$$

The equation

$$\tau_0 = \tau_0 + \frac{0,5}{\tau_u} \quad (4.68)$$

gives an upper limit for the submerged flow. For practical needs it is sufficiently precise to assume that submersion takes place when the downstream water level reaches the point that is designated by the relationship \( \tau_u = \tau_0 = 2,0 \).

Fig. 4.12  Boundaries for the subcritical overflow
A parabolic formula can be found for the transitional area:

$$\tau_0 = (\tau_0)_0 + \frac{2 - (\tau_0)_0}{(2 - (\tau_u)_0)} (\tau_u - (\tau_u)_0)^n$$  \hspace{1cm} (4.69)

This parabolic equation holds for any weir shape, if the discharge coefficient $C$ is known for the free overflow. The different amounts of discharge and the corresponding discharge coefficients require for a set of parabolas (see the following figure).

---

**Fig. 4.13** Diagram for free/subcritical overflow and submerged flow
4.1.3 Weirs with free (critical) overflow

The free overflow at weirs leads to hydraulic independent flow conditions upstream and downstream of the crest because a clear relationship between discharge and water level exists. The upstream water level only depends on the discharge and the weir shape. The most common overflow formulas have been derived by DU BUAT, POLENI and WEISBACH.

DU BUAT:  
\[ Q = \frac{2}{3} \mu b \sqrt{2g \cdot h_{E,o}^3} \]  
(4.70)

POLENI:  
\[ Q = \frac{2}{3} \mu b \sqrt{2g \cdot h_0^{2/3}} \]  
(4.71)

WEISBACH:  
\[ Q = \frac{2}{3} \mu b \sqrt{2g} \left[ h + \frac{v_d^2}{2g} \right] - \left( \frac{v_o^2}{2g} \right)^{3/2} \]  
(4.72)

4.1.3.1 Discharge coefficient for a round-crested weir

Fig. 4.14 Sketch of a weir with free overflow
The crest shape of a round-crested weir corresponds to the lower jet surface of a free, aerated nappe at a sharp-crested weir. This means that neither positive nor negative pressure occurs at the crest and the nappe just clings to the crest. For a crest shaped in this way the discharge coefficient is calculated according to KNAPP (1960) as follows:

$$\mu_a = 0.7825 \left(1 - \zeta \right)^{3/2}$$  \hspace{1cm} (4.73)

with:

$$\zeta = 0.015 \left( \frac{h_{E,a}}{w} \right)^{0.9742} - 0.0326$$  \hspace{1cm} (4.74)

$$h_{E,a} = \frac{h_{E,n}}{1 - \zeta}$$  \hspace{1cm} (4.75)

$$h_{E,n} = \frac{r_w}{0.6879}$$  \hspace{1cm} (4.76)

with:

- $w$ = upstream weir height
- $r_w$ = curvature radius of crest in meters

The discharge coefficient has to be corrected if the expected discharge does not match the real discharge. The discharge coefficient for the range $h_{E,o} < h_{E,a}$ has to be calculated according to KNAPP, and the coefficient for the range $h_{E,o} > h_{E,a}$ according to SCHIRMER.

KNAPP, for the range $h_{E,o} < h_{E,a}$:

$$\mu = 0.5773 \cdot \left(1 + \frac{h_{E,o}}{h_{E,a}}\right)^{3.3219 \log(1.732/h_{E,o})}$$  \hspace{1cm} (4.77)

with:

$$h_{E,o} = h_o + \frac{v_o^2}{2g}$$  \hspace{1cm} (4.78)

SCHIRMER, for the range $h_{E,o} > h_{E,a}$:

$$\mu = 0.9877 \cdot \left[1 - 0.015 \left( \frac{h_{E,o}}{w_o} \right)^{0.9742} \right]^3 \cdot \left[0.8003 + 0.0814 \frac{h_{E,a}}{w_o} - 0.2566 \frac{h_{E,a}}{h_{E,a}} - 0.0822 \frac{h_{E,o}}{h_{E,a}} + 0.0646 \left( \frac{h_{E,a}}{w_o} \right)^2 - 0.0691 \left( \frac{h_{E,a}}{h_{E,a}} \right)^2 + 0.00598 \left( \frac{h_{E,o}}{h_{E,a}} \right)^3\right]$$  \hspace{1cm} (4.79)
4.1.3.2 Discharge coefficient for broad-crested weirs and dam thresholds

Abb. 4.15 Sketch of a broad-crested weir with free overflow

If the crest length is so long that influences of curved stream lines on the discharge capacity of the weir are negligible, the weir is called broad-crested. According to KNAPP this takes place if the following relation is met:

\[
\frac{V^2}{2g} > \frac{1}{3} h_{E,o}
\]

(4.80)

with:

\( l_{we} = \) crest length in direction of flow

Under these flow conditions the pressure is hydrostatic with almost parallel streamlines at the weir body.

According to PETER (1994), the discharge capacity of a broad-crested weir mainly depends on the following properties:

- Weir height \( w_o \)
- Weir elevation above downstream water level \( h_u \)
- Weir length \( l_{we} \)
- Crest shape
- Boundary roughness of the weir body (for weirs in rivers)

For \((l_{we}/h_{E,o}) < 3\) the discharge coefficient is mainly influenced by the weir length and the influence of weir shape, boundary roughness of the weir body and weir height \(w_o\) is negligible. In this case the formula for the discharge coefficient according to KNAPP is as follows:

\[
\mu = 0.61 \cdot \left( \frac{h_{E,o}}{l_{we}} \right)^{0.0544}
\]

(4.81)

For \((l_{we}/h_{E,o}) > 3\) KUMIN has determined the discharge coefficients for different crest shapes depending on the ratio \(h_{E,o}/w\). They are represented in the following diagram.

<table>
<thead>
<tr>
<th>(h_{E,o}/w)</th>
<th>(\alpha)</th>
<th>(r/h_{E,o})</th>
<th>(a/h_{E,o})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((w=0))</td>
<td>90° 45° 22-26°</td>
<td>0.025 0.10 0.4 0.8 1.0</td>
<td>0.025 0.1 2</td>
</tr>
<tr>
<td>5</td>
<td>0.577 0.577 0.577</td>
<td>0.577 0.577</td>
<td>0.577 0.577</td>
</tr>
<tr>
<td>1.67</td>
<td>0.525 0.555 0.570</td>
<td>0.541 0.551 0.561</td>
<td>0.538 0.550</td>
</tr>
<tr>
<td>1.0</td>
<td>0.513 0.551 0.566</td>
<td>0.532 0.543 0.556 0.564</td>
<td>0.529 0.546</td>
</tr>
<tr>
<td>0.5</td>
<td>0.499 0.545 0.564</td>
<td>0.524 0.537 0.552 0.562 0.573</td>
<td>0.520 0.538</td>
</tr>
<tr>
<td>0.167</td>
<td>0.488 0.540 0.562</td>
<td>0.516 0.537 0.549 0.559 0.569</td>
<td>0.511 0.531 0.540</td>
</tr>
<tr>
<td>(w&gt;&gt;h_{E,o})</td>
<td>0.480 0.536 0.561</td>
<td>0.510 0.526 0.546 0.558 0.562</td>
<td>0.505 0.528 0.535</td>
</tr>
</tbody>
</table>

Fig. 4.16 Discharge coefficient \(\mu\) for different crest shapes of broad-crested weirs (from BWK 1999)
The diagram shows that for \( w = 0 \) there is no loss and the discharge coefficient has the largest possible value: \( \mu = \left( \frac{1}{\sqrt{3}} \right) = 0.577 \).

### 4.1.3.3 Discharge coefficient for dam-like weirs

The discharge capacity of weirs of this shape is approximately the same as the expected discharge of a round-crested weir if to the lower nappe surface is sufficiently aerated. It is recommended to use the formula for round-crested weirs using \( h_{E,o} \) for the upstream energy head \( h_{E,o} \).

![Sketch for overflow at a dam-like weir](image)

**Fig. 4.17** Sketch for overflow at a dam-like weir

### 4.1.4 Weirs with subcritical overflow

The overflow is subcritical if the discharge is influenced by a respectively high downstream water level \( h_{uw} \). If the downstream water level rises the critical overflow becomes subcritical. Now the discharge capacity and the loss at the weir do not only depend on the upstream energy head \( h_{E,o} \) but also on the downstream water level. There are boundary conditions for round-crested weirs that define the transition between critical and subcritical overflow. For broad-crested weirs the free overflow immediately becomes subcritical when the critical
depth at the downstream weir side is exceeded (KNAPP, 1969). An evaluation of momentum conditions at the downstream weir side helps to describe this critical state mathematically.

<table>
<thead>
<tr>
<th>Discharge state</th>
<th>Submerged flow</th>
<th>Free overflow</th>
<th>Subcritical overflow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Round-crested</td>
<td>$\frac{h_{uw}}{h_{yr}} \geq 2.0$</td>
<td>$h_{uw} \leq 3,286 - 1,905 \frac{h_{E,gr}}{h_{yr}}$</td>
<td>$2.0 \leq \frac{h_{uw}}{h_{yr}} \leq 3,286 - 1,905 \frac{h_{E,gr}}{h_{yr}}$</td>
</tr>
<tr>
<td>Broad-crested</td>
<td>$h_{uw} \leq \left(1 + \frac{w_{u}}{h_{yr}}\right)^2 + 2 + \frac{1}{g} \left(\frac{v_{u}^2}{h_{yr}} \right) - \frac{v_{u}^2}{g \cdot h_{yr}} - w_{u}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 4.18  Boundary conditions for the different discharge states at weirs

The discharge coefficient for the free overflow has to be corrected with a factor $c_u$ in order to consider the capacity loss caused by the influence of the downstream water level on the overflow, which results in a decrease of the discharge capacity.

$$\mu = c_u \cdot \mu_{volk}$$  \hspace{1cm} (4.82)

$$c_u = \sqrt{1 - \left(\frac{h_{uw}}{h_{u}}\right)^n}$$  \hspace{1cm} (4.83)

with:

$\mu_{volk}$  = Discharge coefficient for free overflow

$n$  = Exponent for different weir crest shapes

<table>
<thead>
<tr>
<th>Crest shape</th>
<th>Exponent $n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Broad-crested</td>
<td>16</td>
</tr>
</tbody>
</table>
### 4.1.5 Submerged weirs (flow over crest)

The flow is considered submerged if the boundary conditions from figure 16 are exceeded. The form drag caused by the weir can be estimated in this case by comparison to flow over a rectangular plate according to NAUDASCHER (1992). The loss at rectangular plates is calculated neglecting wave and riverbed influences:

\[ h_v = c_w \cdot \frac{w \cdot v_o^2}{h_o \cdot 2g} \]  

(4.84)

In the case of a sharp-crested weir body the form drag coefficient has the greatest value of \( c_w = 1.9 \) according to NAUDASCHER (1992). It is lower if the weir shape is more streamlined.
4.1.6 Weirs with different crest heights

If the weir crest has different heights, the weir body has to be partitioned into sections of equal height. Assuming a constant energy height $h_{E,o}$ the partial discharge $q_j$ and the discharge coefficient $\mu_j$ have to be calculated for each section.

$$q_j = \frac{2}{3} \cdot \sqrt{2g \cdot \mu_j \cdot h_{E,o}^{3/2}}$$  \hspace{1cm} (4.85)

The energy height $h_{E,o}$ as well as the partial discharges $q_j$ are unknown. Due to the implicit linking of the partial discharges with the energy height $h_{E,o}$ the calculation can be done in iterations varying $h_{E,o}$ until the total discharge is equal to the real discharge.

$$Q_{ges} = \sum_{j=1}^{n} (q_j \cdot b_{w,j})$$  \hspace{1cm} (4.86)

with:

$n = \text{number of weir sections with constant height}$

$b_{w,j} = \text{width of section } j$

4.2 Hydraulic loss at sluice gates

The discharge at a sluice gate is governed by the degree of flow redirection and the loss due to boundary roughness.
According to the conditions at weirs, there is a difference between free and subcritical discharge at a sluice gate. In the case of free discharge the upstream water level only depends on the discharge and the hydraulic loss at the sluice gate. The downstream water level only has an influence on the upstream water level if the flow is subcritical.

**4.2.1 Sluice gate with free discharge**

According to KNAPP (1960) the total discharge of a sluice gate is calculated as:

\[
q = \zeta_R \cdot \mu_e \cdot a \cdot \sqrt{2g \cdot h_{E,o} - \mu a} \tag{4.87}
\]

with:

- \( \mu_e = \mu \cdot \mu' \)
- \( \mu = \) vertical contraction coefficient
- \( \mu' = \) horizontal contraction coefficient (= 1, if \( b_o = b_u \))
- \( \zeta_R = \) drag coefficient
- \( a = \) gate width
- \( \mu a = \) gate height
- \( h_{E,o} = h_o + \frac{v_o^2}{2g} \)
If the discharge and the empiric coefficients are known, the upstream water level at a sluice with free discharge can be calculated as:

\[ h_o = \left( \frac{q}{\zeta_R \cdot \mu_F \cdot a} \right)^2 \cdot \frac{1-\Phi^2}{2g} + \mu a \]  

(4.88)

with:

\[ \Phi = \zeta_R \cdot \mu_F \cdot \frac{b_o / b_o}{h_o / a} \]  

(4.89)

for a rectangular cross-section.

Fig. 4.22 Sketch of a sluice gate with free discharge

4.2.2 Sluice gate with subcritical discharge

In the case of subcritical discharge the water touches the overlying boundary of the sluice gate and thus influences the discharge conditions at the sluice gate. This state is reached if the following boundary condition is met:
\[
\begin{align*}
&h_u^3 - h_u^2 \left[ 2s \cdot (1 - \varepsilon) \right] \\
&- h_u \left[ (\mu a)^2 - 2s\varepsilon (h_o - \mu a) + 2s \left( \varepsilon (h_o + s) - \frac{s}{2} \right) + \frac{4 \zeta_{SR}^2 \cdot \mu F \cdot a \cdot b_s}{b_{sp,u} (1 - \phi^2)} (h_o - \mu a) \right] \\
&+ \frac{4 \zeta_{SR}^2 \cdot \mu F^2 \cdot (a \cdot b_s)^2}{b_{sp,u}^2 (1 - \phi^2)} (h_o - \mu a) \geq 0
\end{align*}
\] (4.90)

In the case of subcritical discharge the upstream energy height is calculated according to KNAPP (1960) as:

\[
h_{E,u} = h_u + \Delta H
\] (4.91)

with:

\[
\Delta H = - \frac{m}{b_u} \pm \sqrt{\left( \frac{m}{b_u} \right)^2 + \frac{2n}{b_u}}
\] (4.92)

\[
m = \frac{2 \cdot \zeta_{SR} \cdot \mu F \cdot a \cdot b_s}{1 - \phi^2} \left( \zeta_{SR} - X \right) - b_u \cdot (s \cdot \varepsilon + h_o)
\] (4.93)

\[
n = \frac{b_u}{2} (h_u^2 - h_o^2) - s \cdot b_u \left( h_u (1 - \varepsilon) + \varepsilon (h_o + s) - \frac{s}{2} \right)
\] (4.94)

\[
X = \zeta_{SR} \cdot \mu F \cdot \frac{b_s / b_u}{h_u / a}
\] (4.95)

\[
\frac{h_u}{h_u} = \left[ 1 + 2Fr_u^2 \left( 1 - \frac{h_u}{\mu a} \right) + 2s \varepsilon \frac{\Delta H}{h_u^2} + 2s \left( \frac{1 - \varepsilon}{h_u} + \varepsilon \frac{h_o + s}{h_u^3} - \frac{s}{2h_o^2} \right) \right]^{1/2}
\] (4.96)

\[
Fr_u = \frac{Q^2}{gh_u^2 b_u^2}
\] (4.97)

These relationships is valid for sluices with and without lower sill.

### 4.3 Hydraulic loss at local obstacles

Local obstacles in a stream cause an additional loss that is generally not covered by friction loss. Obstacles this applies to are for example:

- Single obstacles like: piers, walls and horizontal bars
- Submerged walls
Bridges, outlets and weirs cause complicated flow states with changing conditions in general, so that the calculation of corresponding losses of pressure height requires differentiated approaches of the problem which are described in chapters 4.1 (weirs), 4.2 (sluice gates) and 4.4 (bridges and outlets). The losses occurring at the other obstacles can be calculated with the following equation:

$$h_{\text{ort}} = \zeta_{\text{ort}} \cdot \frac{v^2}{2g} \quad (4.98)$$

In the following, hints for estimating the loss coefficient are given.

### 4.3.1 Piers, walls, horizontal bars

The loss coefficient for a single obstacle is approximated according to NAUDASCHER (1992) as follows:

$$\zeta_{\text{EH}} = c_w \cdot \frac{A_y}{A_j} \quad (4.99)$$

with:

- $A_j$ = partial cross-sectional area of section j (incl. obstructed cross-section)
- $j =$ left (li) and right (re) flooded plain, river bed (F)
- $A_v = \sum_{j=1}^{n} (d_{v,j} \cdot h_{v,j}) = \text{total area of obstacles}$
- $d_{v,j} \cdot h_{v,j} = \text{area of single obstacle j}$
- $n =$ number of obstacles in section j
- $c_w =$ drag coefficient according to the table in Fig. 4.23

The flow velocity in equation (4.98) refers to the unobstructed cross-section upstream of the obstacle. For bridge piers the pier loss is more precisely calculated with the help of the Rehbock pier formula explained in chapter 4.4.1.
4.3.2 Rakes

With the Kirschmer formula the loss coefficient $\zeta_{SR}$ of the rake can be estimated as:

$$\zeta_{SR} = c_w \left( \frac{d_R}{a} \right)^{4/3} \left( \frac{A_i}{\eta \cdot A_V} \right) \sin \beta \cdot f_k \tag{4.100}$$

with:

- $c_w =$ drag coefficient of rake bars according to table in Fig. 4.24
- $d_R =$ rake bar diameter
- $a =$ rake bar distance
- $A_i =$ flow area width downstream of the rake
- $A_V = \sum_{j=1}^{n} d_{R,j} \cdot h_{R,j} =$ (projected area of the rake for vertical bar rakes)
- $h_{R,j} =$ height of rake bar $j$ in m
- $\eta =$ "rate of permeability" of the rake
- $\beta =$ horizontal angle of rake
- $f_k =$ correctional factor for slanted rakes at angle $\Omega$ according to table in Fig. 4.25
The flow velocity in equation (4.98) refers to the unobstructed cross-section downstream of the rake.

<table>
<thead>
<tr>
<th>$d_R/a$</th>
<th>1.0</th>
<th>0.9</th>
<th>0.8</th>
<th>0.7</th>
<th>0.6</th>
<th>0.5</th>
<th>0.4</th>
<th>0.3</th>
<th>0.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega = 0^\circ$</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>$\Omega = 10^\circ$</td>
<td>1.06</td>
<td>1.07</td>
<td>1.08</td>
<td>1.09</td>
<td>1.10</td>
<td>1.11</td>
<td>1.12</td>
<td>1.13</td>
<td>1.14</td>
</tr>
<tr>
<td>$\Omega = 20^\circ$</td>
<td>1.14</td>
<td>1.16</td>
<td>1.18</td>
<td>1.21</td>
<td>1.24</td>
<td>1.26</td>
<td>1.31</td>
<td>1.43</td>
<td>2.25</td>
</tr>
<tr>
<td>$\Omega = 30^\circ$</td>
<td>1.25</td>
<td>1.28</td>
<td>1.31</td>
<td>1.35</td>
<td>1.44</td>
<td>1.50</td>
<td>1.64</td>
<td>1.90</td>
<td>3.60</td>
</tr>
<tr>
<td>$\Omega = 40^\circ$</td>
<td>1.43</td>
<td>1.48</td>
<td>1.55</td>
<td>1.64</td>
<td>1.75</td>
<td>1.88</td>
<td>2.10</td>
<td>2.56</td>
<td>5.70</td>
</tr>
<tr>
<td>$\Omega = 50^\circ$</td>
<td>1.75</td>
<td>1.85</td>
<td>1.96</td>
<td>2.10</td>
<td>2.30</td>
<td>2.60</td>
<td>3.00</td>
<td>3.90</td>
<td>–</td>
</tr>
<tr>
<td>$\Omega = 60^\circ$</td>
<td>2.25</td>
<td>2.41</td>
<td>2.62</td>
<td>3.26</td>
<td>3.26</td>
<td>3.74</td>
<td>4.40</td>
<td>6.05</td>
<td>–</td>
</tr>
</tbody>
</table>
4.3.3 Submerged wall

According to PRESS/SCHRÖDER (1966) the loss coefficient $\zeta_i$ is calculated as:

$$\zeta_i = \zeta_E - 1 + \frac{h_i^2}{(h_i - h_i)^2} \cdot \frac{2 \cdot \sqrt{1 - 2Fr_i^2 (h_i - h_i) - 1}}{Fr_i^2}$$  \hspace{1cm} (4.101)

$\zeta_E$ = inlet loss
$h_i$ = depth of wall

$Fr_i = \frac{v_i}{\sqrt{g \cdot h_i}}$

The flow velocity in equation (4.98) refers to the unobstructed cross-section downstream of the submerged wall.

4.3.4 Sudden change of cross-section

Based on Borda-Carnot’s approach the inlet and outlet loss for sudden changes of the cross-section is calculated as:

Outlet loss due to widening:

$$\zeta_A = c_a \cdot \left( \frac{A_i}{A_{i+1}} - 1 \right)^2$$  \hspace{1cm} (4.102)

Inlet loss:

$$\zeta_E = c_e \cdot \left( \frac{A_i}{A_{i+1}} - 1 \right)^2$$  \hspace{1cm} (4.103)

The flow velocity in equation (4.98) refers to the underwater profile i in both cases.
The coefficients $c_a$ and $c_e$ can be approximated in dependence of the obstacle contour according to Fig. 4.26.
4.3.5 Sudden widening at sill

The loss coefficient for the sudden widening of the cross-section due to a negative step in the bed for a rectangular profile is calculated according to BOLLRICH (1996) as follows:

\[
\zeta_s = 1 - \left( \frac{h_{i+1}}{h_i} \right)^2 + \frac{2 \left( 1 - \frac{h_i}{h_{i+1}} + \frac{s}{h_{i+1}} \right)}{Fr_{i+1}^2}
\]  

(4.104)

with:

\[ s = \text{height of the negative step} \]

\[ Fr_{i+1} = \frac{v_{i+1}}{\sqrt{g \cdot h_{i+1}}} \]  

(4.105)

The index \( i \) refers to the cross-section below and the index \( i+1 \) to the cross-section above the step.

Furthermore the flow velocity in equation (4.98) refers to the downstream cross-section.
4.4 Hydraulic loss at bridges and outlets

Vertical and horizontal constrictions can cause an additional hydraulic loss at bridges and outlets that not only leads to local disturbances in the flow, but significantly determines the water level especially upstream of a bridge or a similar construction. A calculation of the occurring hydraulic loss using the water line equation does not yield satisfactory results. In dependence of the flow state at the bridge the calculation has to be more differentiated. The following 5 discharge states as shown in Fig. 4.27 are most relevant and methods for estimating the hydraulic loss for each state are presented in the following.

![Discharge states at bridges](image)

**Fig. 4.27**  Discharge states at bridges

4.4.1 Free discharge at a bridge

The discharge is free if the water level upstream as well as downstream of the bridge is below the lower edge of the bridge. Horizontal narrowing due to bearings, access ramps or piers can cause flow resistance. With the Rehbock pier formula the flow resistance of a pier can be calculated differentiating a compact cross-section with an almost even flow velocity and a structured cross-section with an uneven velocity distribution. The Rehbock pier formula can also be used to calculate the loss as a result of side-constructions (based on model...
experiments by SCHWARZE (1969)). The calculated backwater has to be multiplied with a correctional factor.

\[
\Delta h = c \cdot \Delta h_{\text{Rehbock}}
\]  
(4.106)

The coefficient \(c\) takes the following values depending on the channel shape (Fig. 27)

<table>
<thead>
<tr>
<th>River bed geometry</th>
<th>(c)</th>
<th>Valid range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>0,4 + 2,45 (A_{\text{verb}} / A_u)</td>
<td>(A_{\text{verb}} / A_u \leq 0,24)</td>
</tr>
<tr>
<td>Trapezoidal</td>
<td>2</td>
<td>(0,2 + 2,33 \cdot A_{\text{verb}} / A_u)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>(0,28 \cdot A_{\text{verb}} / A_u)</td>
</tr>
</tbody>
</table>

Fig. 4.28 Coefficient \(c\) for horizontal narrowing at a bridge

However, SCHWARZE’s method is valid only for minor horizontal narrowing, with boundary criteria for \(A_{\text{verb}} / A_u\) (relationship between obstructed cross-sectional area and total cross-section at the lower water level) as shown in the table.

If these boundary criteria are exceeded the hydraulic, loss as a result of horizontal narrowing has to be approximated as shown in chapter 4.3 with approach (4.98) and loss coefficients \(\zeta\) according to equations (4.102) and (4.103).

Three states of discharge for pier backwater calculations and compact cross-sections are differentiated: subcritical flow, supercritical flow with wavy water surface and supercritical flow with a hydraulic jump. The pier backwater is calculated for each discharge state as shown in the table in Fig. 4.29 using the shape coefficient \(\delta = \delta_o + A_{\text{verb}} / A_u \cdot \delta_o\), the Froude-number \(Fr = \frac{v_b}{\sqrt{g \cdot h_u}}\); and the mean flow velocity of the bridge cross-section \(v_b\).

\(\delta_o = \) as in Fig. 4.30

<table>
<thead>
<tr>
<th>Flow state</th>
<th>Boundary criteria</th>
<th>(\Delta h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subcritical flow</td>
<td>(A_{\text{verb}} / A_u \leq \frac{1}{(0,97 + 10,5Fr^2)} - 0,13)</td>
<td>(\delta \left(0,4A_{\text{verb}} / A_u + \left(A_{\text{verb}} / A_u\right)^2\right)^\frac{1}{3} + 9\left(A_{\text{verb}} / A_u\right)^4 + 8 \cdot \left(1 + Fr^2\right) \cdot \frac{v_b^2}{2g})</td>
</tr>
</tbody>
</table>
Flow with wavy water surface

\[ A_{\text{verb}} / A_u \leq 0.05 + \left( 0.9 - 1.25 Fr^2 \right)^2 \]

\[ \left( 21.5 A_{\text{verb}} / A_u + 15.5 Fr^2 - 6.6 \right) \cdot \frac{v_j^2}{2g} \]

Supercritical flow with hydraulic jump

\[ A_{\text{verb}} / A_u > 0.05 + \left( 0.9 - 1.25 Fr^2 \right)^2 \]

Application of the extremal principle

---

**Fig. 4.29** Pier backwater formulas for compact bridge cross-sections

<table>
<thead>
<tr>
<th>Pier shape</th>
<th>( \delta_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>rectangular</td>
<td>[ l_{pf} &gt; 13.5 \text{ m} ] [ 3.1 + 0.12 \cdot \frac{l_{pf}}{d_{pf}} ]</td>
</tr>
<tr>
<td></td>
<td>[ l_{pf} \leq 13.5 \text{ m} ] [ 4.85 + 0.27 \cdot \frac{l_{pf}}{d_{pf}} ]</td>
</tr>
<tr>
<td>slanted noses, both sides</td>
<td>[ 2.87 + (1.01 - 1.74 \varepsilon)^3 ]</td>
</tr>
<tr>
<td>sharp noses, both sides</td>
<td>[ 1/\left( 0.206 + 0.116 \cdot l_k / d_{pf} \right) - 0.95 ]</td>
</tr>
<tr>
<td>semicircular noses, both sides</td>
<td>[ 1 + \left( 1.04 - 0.35 \cdot \frac{l_k}{d_{pf}} \right)^2 + \sqrt{\left( \frac{9 \cdot l_k}{d_{pf}} \right)^4 + 0.74} ]</td>
</tr>
</tbody>
</table>

\( l_{pf} \) pier length in direction of flow

\( d_{pf} \) pier width

\( l_k \) noses length

---

**Fig. 4.30** Pier shape coefficient \( \delta_0 \) (according to SCHWARZE (1969))

For structured cross-sections with uneven velocity distribution SCHWARZE (1969) suggested to correct the pier backwater with a factor, similar to horizontal contractions.

\[ \Delta h = c \cdot \Delta h_{\text{Rehbook}} \] (4.107)

The correctional factor is computed as shown in Fig. 4.31, with \( A_{pf,j} = d_{pf,j} \cdot h_j \) = obstructed cross-section, \( A \cdot v = Q_{\text{ges}} \), \( v_j \) = velocity upstream of the bridge and the obstructed section \( j \).
<table>
<thead>
<tr>
<th>Geometry of section</th>
<th>c</th>
<th>$a_Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>0.68 + $2a_Q$</td>
<td></td>
</tr>
<tr>
<td>Trapezoidal</td>
<td>$m = 2$</td>
<td>0.47 + $2a_Q$</td>
</tr>
<tr>
<td>Bank slope</td>
<td>$m = 3$</td>
<td>0.30 + $2a_Q$</td>
</tr>
</tbody>
</table>

$$A_v = \sum_{j=1}^{n} \frac{(A_{pi} \cdot v_j)}{A \cdot v}$$

**Fig. 4.31** Correctional factor for pier backwater in structured cross-sections with uneven velocity distribution

**Fig. 4.32** Sketch for discharge situation at bridges with partitioned cross-section
4.4.2 Dammed-in bridge structure with free discharge under the bridge

The hydraulic loss and the upstream water level at the bridge are calculated with the help of equation (4.88) since the discharge state corresponds to free discharge at a sluice gate with slope \( \alpha = 90^\circ \) and generally sill height \( s = 0 \).

However, the coefficient \( \zeta_R \) in the equation for the hydraulic loss due to friction and flow diffusion upstream of the sluice gate only accounts for almost smooth surfaces like concrete walls. The bed and bank structure of bridge structures generally do not meet this criterion.

It is suggested to consider the hydraulic loss in discharge calculations for bridges even if the friction loss is generally much smaller compared to other hydraulic losses at a dammed-in bridge. On the one hand the numerical effort is small and on the other hand the friction loss will not be neglected in boundary situations with extreme bed roughness.

\[ h_o = \left( \frac{Q}{\zeta_R \cdot \mu \cdot a \cdot b_{Br}} \right) \cdot \frac{1 - \Phi^2}{2g} + \mu \cdot a + h_i \]  

(4.108)

with:

\[ h_b = \frac{\lambda_b \cdot l_b \cdot V_b^2}{4r_{hy,b} \cdot 2g} \]  

(4.109)

\( l_b = \) length of bridge in direction of flow

\( r_{hy,b} = \frac{A_b}{l_{ub}} \)
\( \lambda_b = \) flow resistance due to boundary roughness at the bridge

The parameter \( \zeta_R \) in equation (4.108) has to be corrected in such a way that it not only accounts for a hydraulic loss because of stream redirection, but also considers the hydraulic loss because of the bed-roughness.

However, this can only be approximated with today’s knowledge.

With the resistance laws from chapter 3.4 the flow resistance \( \lambda_b \) due to boundary roughness can be calculated.

A contraction loss becomes important if extensive horizontal contraction prevails. It is obtained according to BORDA-CARNOT as explained in chapter 4.3.4. This leads to an extension of equation (4.108) by the loss head \( h_{\text{Verbau}} \) that is calculated as follows:

\[
h_{\text{Verbau}} = \zeta_{\text{uu}} \cdot \frac{v_u^2}{2g} + \zeta_{\text{ov}} \cdot \frac{v_o^2}{2g}
\]

with:

\[
\zeta_{\text{uu}} = c_a \left( \frac{A_o}{A_u} - 1 \right)^2
\]

\[
\zeta_{\text{ov}} = c_a' \left( \frac{A_o}{A_v} - 1 \right)^2
\]

\( c_a, c_a' \) values according to Fig. 4.26

4.4.3 Dammed-in bridge structure with backwater (subcritical discharge)

The hydraulic loss that has to be considered is obtained in analogy to the discharge at a sluice gate with subcritical discharge with the help of equation (4.91) in chapter 4.2.2 since the two discharge situations are almost equal.

For consideration of higher friction loss and hydraulic loss because of horizontal contraction the equation (4.91) (like for a dammed-in bridge with free discharge) has to be extended by the loss heads \( \Delta h_R \) and \( \Delta h_{\text{Verbau}} \).

In general the bed at bridges is even so that the hydraulic loss of a sill below the water surface can be neglected and the equations for the parameters \( m \) and \( n \) become much easier since the sill height is set to \( s = 0 \) and \( \varepsilon = 0 \).
4.4.4 Submerged bridge structure with supercritical flow

The prevalent discharge situation can be divided into two components. On the one hand the stream still discharges through the bridge (pressure discharge) and on the other hand there is a part that is discharged according to a weir over the bridge deck.

The first case mostly corresponds to the case of a sluice gate with submerged flow (corresponds to discharge situation at a dammed-in bridge with submerged flow).

This means the discharge through the bridge can be determined as shown in chapter 4.4.3 without having to take care of the discharge over the bridge.

Fig. 4.34 Sketch of submerged bridge structure with supercritical flow

The relative crest height \( \left( h_{E,o} - h_{bok} \right)/l_b < 3.0 \) generally fits to the spill situation at the bridge. Since this spill situation corresponds to a broad-crested weir, the discharge part over the bridge can be calculated with the DU BUAT equation:

\[
Q_u = \frac{2}{3} \mu b_l \cdot \sqrt{2g} \cdot \left( h_{E,o} - h_{bok} \right)^{3/2}
\]

with the discharge coefficient \( \mu \):

\[
\mu = 0.61 \cdot \left( \frac{h_{E,o} - h_{bok}}{l_b} \right)^{0.0544}
\]

Even if the spill situation mostly corresponds to a broad-crested weir, the discharge coefficients found in Fig. 4.16 (occurring for stream-lined flow edges) can not be assumed since for both upstream and downstream the bridge deck is generally sharp-edged.
Since the discharge partition of submerged and subcritical parts of the bridge depends on the energy head upstream of the bridge and the total discharge, the discharge partition has to be found iteratively. The right partition is found when the continuity equation is met as an additional boundary condition:

$$Q_{ges} = Q_D + Q_U$$

(4.113)

with:

- $Q_D = \text{discharge under the bridge}$
- $Q_U = \text{discharge over the bridge}$

Here $Q_D$ has to be obtained as follows depending on the discharge state:

For free (critical) discharge: with equation (4.108) solved for $Q$

For subcritical discharge:

$$Q_D = \rho_R \cdot \mu_F \cdot a \cdot b_s \sqrt{2g \frac{\Delta H}{1 - \Phi}}$$

(4.114)

### 4.4.5 Dammed-in bridge structure with subcritical overflow

The discharge situation for a dammed-in bridge structure with subcritical overflow is mostly the same as for a dammed-in bridge structure with free overflow. However, the boundary depth above the bridge structure is not met.

According to KNAPP (1960) discharge situations like these occur if the dimensionless downstream crest depth $h_u^*$ meets the flow conditions for a broad-crested weir as pictured in Fig. 4.18.

The definition of $h_u^*$ is:

$$h_u^* = \frac{h_u - h_{bok}}{h_{yr}}$$

(4.115)
Fig. 4.35  Sketch of discharge situation at a submerged bridge with subcritical overflow

The discharge part over the bridge is calculated in analogy to chapter 4.1.5 by the approach for a rectangular plate.

\[ h_v = c_w \cdot \frac{w_{\bar{u}} \cdot v_{\bar{u}}^2}{(h_{\bar{u}} + w_{\bar{u}}) \cdot 2g} \]  \hspace{1cm} (4.116)

with:

\[ v_{\bar{u}} = \frac{Q_{\bar{u}}}{b_b (h_{\bar{u}} + w_{\bar{u}})} \]  \hspace{1cm} (4.117)

\[ w_{\bar{u}} = \frac{Q_{\bar{u}}}{Q_{ges}} \cdot w \]  \hspace{1cm} (4.118)

\( w_{\bar{u}} = \) part of bridge deck hit by flow

\( c_w = 1.9 \)

Because of the implicit connection of the parameters in the equations the discharge partition of \( Q_{\bar{u}} \) and \( Q_{\bar{u}} \) has to be determined iteratively. The right discharge partition is found if the continuity equation (4.113) is met on one side and the energy head \( h_{E,o} \) upstream of the...
bridge (equations (4.91) to (4.97)) corresponds to the energy head used in the calculation of the partial discharge $Q_0$ over the bridge structure on the other side of the equation.

$$h_{E,o} = h_{E,u} + h_v$$

with:

$h_v$ according to (4.116)