2.2 TWO-DIMENSIONAL FLOW CALCULATION

2.2.1 BASIC HYDRODYNAMIC EQUATIONS

In the following chapters, the basic hydrodynamic equations for two-dimensional depth-averaged flow calculation will be derived step by step. We will begin with the two-dimensional Navier-Stokes equations for incompressible fluids, commence with Reynolds equations (time-averaged), and end with the depth-averaged shallow water equations.

2.2.1.1 DERIVATION OF THE NAVIER-STOKES EQUATIONS

The derivation of the Navier-Stokes equations is closely related to [Schlichting et al., 1997], that carries out the derivation in detail.

2.2.1.1.1 MAIN THOUGHTS, DESCRIPTION OF FLOW FIELDS

In three-dimensional movement, the flow field is mainly determined by the velocity vector:

\[
\vec{v} = \vec{e}_x u + \vec{e}_y v + \vec{e}_z w
\]

with \( u, v, w: \) velocity components,

\( \vec{e}_x, \vec{e}_y, \vec{e}_z: \) unit vectors,

but also by the pressure \( p \) and the temperature \( T \).

These 5 variables principally change with time and place:

\[
\vec{v} = \vec{v}(x, y, z, t) \\
p = p(x, y, z, t) \\
T = T(x, y, z, t)
\]

Eq. 2-85

There are five equations available when determining these five variables:

- Continuity equation (preservation of mass).
- 3 momentum / motion equations (preservation of momentum).
- Energy equation (preservation of energy, 1. theorem of thermodynamics).

For the isotropic Newton fluids that are regarded here, the 5 balance equations contain the following material variables that generally depend on temperature and pressure:

- Density \( \rho \).
- Isobaric specific heat capacity \( c_p \).
- Viscosity \( \mu \).
- Heat conductivity \( \lambda \).
2.2.1.1.2 PRESERVATION LAWS OF MASS, MOMENTUM AND ENERGY

2.2.1.1.2.1 Continuity equation
The continuity equation states the preservation of mass. It also expresses that the sum of mass flowing in and out of a volume unit per time is equal to the change of mass per time divided by the change of density [Schlichting et al., 1997].

This yields for the unsteady flow of a general fluid:

\[
\frac{D\rho}{Dt} + \rho \cdot \text{div} \, \mathbf{\bar{v}} = 0 \quad \text{Eq. 2-86}
\]

or written in a different way:

\[
\frac{\partial \rho}{\partial t} + \text{div} (\rho \, \mathbf{\bar{v}}) = 0 \quad \text{Eq. 2-87}
\]

This formula is derived with Euler’s or Lagrange’s approach. The difference between the two approaches shall be briefly shown here. Stream mechanics pulls its knowledge from the observation of processes in nature. If one chooses a fix place to observe a flow, this is the Eulerian point of view. The hydrodynamic-numerical simulation models make use of this way of observation in order to be able to calculate variables like velocity, density or pressure in a point depending on time. Mathematically this expresses in that we only need the partial derivative \( \frac{\partial f}{\partial t} \) to describe the variance in time. However, one only gets knowledge about the value of a certain variable at a certain place in this way; not, however, about the value of this variable in a certain area. This requires having a formula for the place of the particle as a function of time, which is the Lagrange point of view. The observer so to say moves with the fluid particle. In order to describe the variance in time of the moving particle, the trajectory derivative \( \frac{Df}{Dt} \), i.e. the total derivative, is needed in this case. In theory, the Lagrange point of view makes sense, but practice shows that the tracing of a single fluid particle is very difficult.

![Figure 2-11: Eulerian point of view.](image)
The Eularian point of view, that needs a fixed volume, describes an increase of mass over the bounds by volume flow and is expressed mathematically as:

\[ \frac{d\text{m}(t)}{dt} = \frac{d}{dt} \int \rho dV = -\oint \rho \cdot \vec{\nu} \cdot d\vec{A} \]  
Eq. 2-88

In case the volume flow increases the mass, this opposes the positive direction (see drawing), so there is a minus sign on the right side.

If we look at the terms separately, we get:

**Left side:**

\[ \frac{d}{dt} \left( \rho dV \right) = \frac{\partial}{\partial t} (\rho dV) = \oint \frac{\partial \rho}{\partial t} dV \]  
Eq. 2-89

Since the boundaries are fixed and do not change in time, they can be exchanged.

**Right side:**

The surface integral is transcribed into a volume integral:

\[ -\oint \rho \cdot \vec{\nu} d\vec{A} = -\int \text{div} (\rho \vec{\nu}) dV \]  
Eq. 2-90

If we now set the two results equal, we get:

\[ \oint \frac{\partial \rho}{\partial t} dV = -\int \text{div} (\rho \vec{\nu}) dV \]  
Eq. 2-91

Eq. 2-91 integrated is:

\[ \frac{\partial \rho}{\partial t} + \text{div} (\rho \vec{\nu}) = 0 \]  
Eq. 2-92

because

\[ \text{div} (\rho \vec{\nu}) = \vec{\nu} \cdot \text{grad} \rho + \rho \cdot \text{div} \vec{\nu} \]  
Eq. 2-93

is obtained when Eq. 2-93 is substituted into in Eq. 2-92:

\[ \frac{\partial \rho}{\partial t} + \vec{\nu} \cdot \text{grad} \rho + \rho \cdot \text{div} \vec{\nu} = 0 \]  
Eq. 2-94

with \( \text{div} \vec{\nu} = \frac{\partial u}{\partial x} \hat{e}_x + \frac{\partial v}{\partial y} \hat{e}_y + \frac{\partial w}{\partial z} \hat{e}_z \)
The first two terms are the substantial derivative of the density over time. It is composed of the local part for unsteady flows and the convective part due to change in place.

This yields the continuity equation:

\[
\frac{Dp}{Dt} + \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0
\]

Eq. 2-95

We now assume an incompressible fluid\(^1\). The first term of Eq. 2-95, that is the substantial derivative of the density over time, is zero under this precondition. The result of the continuity equation is then that incompressible fluids are solenoidal and the continuity equation can be written as:

\[
div \vec{v} = 0 \quad \text{or} \quad \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0
\]

Eq. 2-96

The constant density in the flow field is a commensurate, but not a necessary condition for incompressible flows.

2.2.1.1.2.2 Momentum / motion equations

The momentum equation is a basic law of mechanics. It states that mass times acceleration is equal to the sum of forces that act on a volume unit. We differentiate between mass forces (weight forces) and surface forces (pressure and friction forces).

\[
dI = dm(t) \cdot \vec{v}
\]

\[
\frac{1}{dV} \frac{dI}{dt} = \frac{1}{dV} \frac{d}{dt} \int \rho \cdot \vec{v} \, dV = \vec{F} + \vec{P}
\]

Eq. 2-97

\(\vec{F}\): mass force per volume unit

\(\vec{P}\): surface force per volume unit

With further simplification we get:

\(^1\) A fluid, which does not change it's density at the pressure from outside, is called incompressible. In nature, all fluids are always compressible. For most calculations the fluid may still be granted as an incompressible fluid, as the error is negligibly small. Moreover, this assumption simplifies the calculation enormously. In the technical hydraulics the compressibility, for example, of a hydraulic fluid should be kept clearly in mind.
\[ \frac{1}{dV} \frac{d}{dt} \int \rho \cdot \vec{v} \, dV = \frac{1}{dV} \int \frac{D}{Dt} (\rho \cdot \vec{v}) \, dV \]

\[ \frac{1}{dV} \frac{D}{Dt} (\rho \cdot \vec{v} \, dV) = \frac{1}{dV} \left( \vec{v} \frac{D}{Dt}(\rho \, dV) + \rho \, dV \frac{D\vec{v}}{Dt} \right) \]

\[ = \rho \frac{D\vec{v}}{Dt} \]

The last expression contains the substantial acceleration that is composed of the local and the convective acceleration:

\[ \frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \frac{\partial}{\partial t} (\vec{v} \cdot \text{grad}) \vec{v} \]

This equation makes the transition from the Lagrange point of view, that has no reference to space, but only regards changes in time, to the Eularian point of view, which accounts for the translation part by convective transport. The momentum equation with these changes is:

\[ \rho \frac{D\vec{v}}{Dt} = \rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \text{grad}) \vec{v} \right) = \vec{F} + \vec{P} \]

The mass forces are to be seen as given, outside, forces, in opposition to the surface forces that depend on the deformation state of the fluid. The total of surface forces makes a stress state. Now a connection between the stress state and the deformation state has to be made. This is done with help of the transport equation.

All further considerations will be limited to isotropic Newton fluids. All gases and many liquids, especially water, belong to this category. If the relation between all components of the stress tensor and the deformation velocity tensor is the same for all directions, the fluid is considered isotropic. If this relation is linear it is considered a Newton fluid. The equation for the momentum transport is called the Newton or Stokes friction law in this case.
2.2.1.1.3 GENERAL STRESS STATE OF DEFORMABLE BODIES

The volume element \( dV = dx \, dy \, dz \) is described as shown in Figure 2-12: Volume element with its stresses. for the determination of the surface forces. The two following resulting stresses act on the planes normal to the x-axis, having the size \( d\Delta = dy \, dz \): 

\[
\sigma_x \quad \text{and} \quad \sigma_x + \frac{\partial \sigma_x}{\partial x} \, dx
\]

Eq. 2-101

This is the same for the other planes. The components of the resulting surface stresses in direction of the three coordinates are as follows:

- Plane \( \perp \) in x-direction: \( \frac{\partial \sigma_x}{\partial x} \cdot dx \cdot dy \cdot dz \)
- Plane \( \perp \) in y-direction: \( \frac{\partial \sigma_y}{\partial y} \cdot dx \cdot dy \cdot dz \)
- Plane \( \perp \) in z-direction: \( \frac{\partial \sigma_z}{\partial z} \cdot dx \cdot dy \cdot dz \)

Eq. 2-102

The total surface force \( \vec{P} \) per volume unit \( dV \) resulting from the stress state is thus:

\[
\vec{P} = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_z}{\partial z}
\]

Eq. 2-103

Here \( \vec{p}_x, \vec{p}_y, \text{ and } \vec{p}_z \) are vectors that can be decomposed. The components orthogonal to the area element, that is the normal stresses, are identified by \( \sigma_i \), the index standing for the direction of the normal stress. The components in the plane of the area element are called tangential stresses \( \tau_{ij} \). The
first index stands for the axis the area element is orthogonal to, the second index for the direction the stress is pointing to. We thus get:

\[
\begin{align*}
\bar{p}_x &= \sigma_x \cdot \hat{e}_x + \tau_{xy} \cdot \hat{e}_y + \tau_{xz} \cdot \hat{e}_z \\
\bar{p}_y &= \tau_{yx} \cdot \hat{e}_x + \sigma_y \cdot \hat{e}_y + \tau_{yz} \cdot \hat{e}_z \\
\bar{p}_z &= \tau_{zx} \cdot \hat{e}_x + \tau_{zy} \cdot \hat{e}_y + \sigma_z \cdot \hat{e}_z
\end{align*}
\]

This results in a stress tensor with nine variables:

\[
\Pi = \begin{bmatrix}
\sigma_x & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z
\end{bmatrix}
\]

Eq. 2-105

This construct is called stress matrix. The stress matrix and its stress tensors are symmetric. This implies:

\[
\begin{align*}
\tau_{xy} &= \tau_{yx} \\
\tau_{xz} &= \tau_{zx} \\
\tau_{yz} &= \tau_{zy}
\end{align*}
\]

Eq. 2-106

The result is a stress matrix that only has six components and is symmetric to the first diagonal:

\[
\Pi = \begin{bmatrix}
\sigma_x & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z
\end{bmatrix}
\]

Eq. 2-107

Eq. 2-103 and Eq. 2-104 yield the surface force per volume unit:

\[
\bar{P} = \hat{e}_x \left( \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) + \hat{e}_y \left( \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) + \hat{e}_z \left( \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} \right)
\]

x – Komp. 
y – Komp. 
z – Komp.

Eq. 2-108
This expression is substituted into Eq. 2-100 and we get:

\[
\rho \frac{Du}{Dt} = F_i + \frac{\partial}{\partial x} \sigma_{xx} + \frac{\partial}{\partial y} \tau_{xy} + \frac{\partial}{\partial z} \tau_{xz} \\
\rho \frac{Dv}{Dt} = F_y + \frac{\partial}{\partial x} \tau_{yx} + \frac{\partial}{\partial y} \sigma_y + \frac{\partial}{\partial z} \tau_{yz} \\
\rho \frac{Dw}{Dt} = F_z + \frac{\partial}{\partial x} \tau_{xz} + \frac{\partial}{\partial y} \tau_{yz} + \frac{\partial}{\partial z} \sigma_z
\]

Eq. 2-109

If the stress state is hydrostatic, that means \( \vec{v} = 0 \), there are no tangential stresses, but only normal stresses that all have equal values. This is the case for unmoving fluids and ideal fluids. We get:

\[
\sigma_x = \sigma_y = \sigma_z = -p
\]

Eq. 2-110

It provides useful to separate the pressure from the normal stress:

\[
\tau_{xx} = \sigma_x + p, \quad \tau_{yx} = \sigma_y + p, \quad \tau_{xz} = \sigma_z + p
\]

Eq. 2-111

The stresses are thus decomposed into a sum of normal stresses \( p \) equal to all sides and an addendum that deviates from it (deviator stresses).

With help of this decomposition, the momentum equation can be written as:

\[
\rho \frac{Du}{Dt} = F_i + \frac{\partial}{\partial x} \left( \sigma_{xx} + \frac{\partial}{\partial y} \tau_{xy} + \frac{\partial}{\partial z} \tau_{xz} \right) \\
\rho \frac{Dv}{Dt} = F_y + \frac{\partial}{\partial x} \left( \tau_{yx} + \frac{\partial}{\partial y} \sigma_y + \frac{\partial}{\partial z} \tau_{yz} \right) \\
\rho \frac{Dw}{Dt} = F_z + \frac{\partial}{\partial x} \left( \tau_{xz} + \frac{\partial}{\partial y} \tau_{yz} + \frac{\partial}{\partial z} \sigma_z \right)
\]

Eq. 2-112

or in vector form:

\[
\rho \frac{D\vec{\tau}}{Dt} = \vec{F} - \text{grad } p + \text{Div } \tau
\]

Eq. 2-113

Here, \( \tau \) is the viscous stress tensor. It only contains the deviator stresses and is symmetric. It is defined as:

\[
\Pi = \begin{bmatrix}
\tau_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{xy} & \tau_{yy} & \tau_{yz} \\
\tau_{xz} & \tau_{yz} & \tau_{zz}
\end{bmatrix}
\]

Eq. 2-114
### 2.2.1.4 General Deformation State of Subcritical Fluids

This chapter will describe the corresponding situation of two particles in a subcritical fluid. We look at a fluid element in the flow that is transferred to a different place over time. Due to the flow field, which is described with help of the velocity vector \( \vec{v} = \vec{v}(x, y, z, t) \), the fluid element is not only subject to a translation, but also to a deformation. This deformation is a function of the velocity field, which has to be shown in the following.

The deformation velocity of a fluid element depends on the relative velocity of two of its points. We assume two adjacent points A and B. Point A will be transferred to point A' in the time \( dt \) due to the velocity field, where \( s = \vec{v}dt \).

**Figure 2-13: Translation of section AB to section A'B'**.

If the velocity in point A is \( \vec{v} = (u, v, w) \), the velocity in point B can be written as:

\[
\begin{align*}
  u + du &= u + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \\
  v + dv &= v + \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz \\
  w + dw &= w + \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz
\end{align*}
\]

**Eq. 2-115**

The relative velocity of point B with respect to A is described by the following matrix of the nine partial derivatives of the local velocity field:

\[
\begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}
\end{pmatrix}
\]

**Eq. 2-116**

In order to continue Eq. 2-115 we introduce the following form for the relative velocity components. It will provide to be useful, similar to the separation of the pressure from the normal stress in Eq. 2-111.
\[
\begin{align*}
\dot{u} &= (\dot{\epsilon}_x \, dx + \dot{\epsilon}_y \, dy + \dot{\epsilon}_z \, dz) + (\omega_x \, dz - \omega_y \, dy) \tag{Eq. 2-117} \\
\dot{v} &= (\dot{\epsilon}_y \, dx + \dot{\epsilon}_z \, dy + \dot{\epsilon}_x \, dz) + (\omega_y \, dx - \omega_x \, dz) \\
\dot{w} &= (\dot{\epsilon}_z \, dx + \dot{\epsilon}_x \, dy + \dot{\epsilon}_y \, dz) + (\omega_z \, dy - \omega_y \, dx)
\end{align*}
\]

with the newly introduced variables:

\[
\dot{\epsilon} = \begin{pmatrix}
\dot{\epsilon}_x \\
\dot{\epsilon}_y \\
\dot{\epsilon}_z \\
\end{pmatrix}
\tag{Eq. 2-118}
\]

\[
\dot{\epsilon} = \begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\
\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\
\frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z}
\end{pmatrix}
\tag{Eq. 2-119}
\]

Tensor of the deformation rate

and with the rotations:

\[
\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \quad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \quad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)
\tag{Eq. 2-120}
\]

The matrix \( \dot{\epsilon} \) is symmetric, so that we can state:

\[
\dot{\epsilon}_{xy} = \dot{\epsilon}_{yx}, \quad \dot{\epsilon}_{xz} = \dot{\epsilon}_{zx}, \quad \dot{\epsilon}_{yz} = \dot{\epsilon}_{zy}
\tag{Eq. 2-121}
\]

\( \omega_x, \omega_y, \omega_z \) are the components of the rotation vector \( \omega = \frac{1}{2} \text{rot} \, \vec{v} \).

At first sight, the form of Eq. 2-117 with the new variables \( \dot{\epsilon} \) and \( \omega \) will seem a little unmotivated, but it can be shown that each of these variables has a kinematical meaning, speaking that Eq. 2-117 has a physical foundation [Schlichting et al., 1997].

If we look at the processes that a fluid element in the velocity field is prone to, we can make a subdivision into:

1. Just the translation, represented by \( u, v, w \).
2. A body rotation, represented by \( \omega_x, \omega_y, \omega_z \).
3. A relative volume change (volume dilatation), represented by the components of the main diagonal \( \dot{\epsilon}_x, \dot{\epsilon}_y, \dot{\epsilon}_z \).
4. A deformation, represented by the three components \( \dot{\epsilon}_{xy}, \dot{\epsilon}_{xz}, \dot{\epsilon}_{yz} \).

The first two points can be referred to as displacements, while the latter two are deformations.
### 2.2.1.1.5 Relationship between Stresses and Deformation Velocity

Now it shall be made clear, that the only way of making a connection between the stresses that act on a fluid element and the deformation velocity is by evaluating empirical physical experiments.

- We consequently only assume isotropic Newton fluids\(^2\).
- We further assume that the viscous stress tensor (Eq. 2-114) only depends on the deformation velocity tensor (Eq. 2-119), that means \( \tau_{ij} \) can be expressed as a function of the velocity gradients. The translation and the body rotation thus do not cause any surface forces.
- The functional relationship between \( \tau_{ij} \) and the velocity gradients is assumed to be linear and is independent from a rotation of the coordinate system and a change of the axis (isotropy\(^3\)).

With these preconditions we can derive the following relationship:

\[
\begin{align*}
\tau_{xx} &= \lambda \text{div} \vec{v} + 2\mu \frac{\partial u}{\partial x} \\
\tau_{yy} &= \lambda \text{div} \vec{v} + 2\mu \frac{\partial v}{\partial y} \\
\tau_{zz} &= \lambda \text{div} \vec{v} + 2\mu \frac{\partial w}{\partial z}
\end{align*}
\]

and

\[
\begin{align*}
\tau_{xy} &= \tau_{yx} = \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \\
\tau_{zx} &= \tau_{xz} = \mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \\
\tau_{yz} &= \tau_{zy} = \mu \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)
\end{align*}
\]

Eq. 2-122

Eq. 2-123

The factors of proportionality \( \mu \) and \( \lambda \) must have the same value for every direction because of the assumption of isotropy. If the equations are applied to simple flows like for example a Couette-flow, the equations are reduced to Newton’s friction law.

\[
\tau = \mu \frac{du}{dy}
\]

Eq. 2-124

In this connection \( u \) is the flow velocity parallel to the wall and \( y \) is the coordinate normal to the wall. The proportionality constant \( \eta \) is described as dynamic viscosity. This way it can be seen that the factor \( \mu \) is equal to the viscosity of the fluid. The factor \( \lambda \) is only of importance if we look at compressible fluids, since \( \text{div} \vec{v} \) is zero for incompressible fluids.

---

\(^2\) A fluid whose shear stress \( \tau \) is proportional to the distortion- and shear speed, is a Newton's fluid (after Isaac Newton). Most of the fluids (e.g. water, air, lots of oils and gases) behave in this sense. Notwithstanding behave non-Newtonian fluids such as blood, glycerine or dough with a non-proportional, erratic flow behaviour. [wikipedia]

\(^3\) Isotropy (greek: isos = equally; greek: tropos = rotation, direction) is the independence of a property from the direction. For example, with a radiation isotropic is meant a radiation that is evenly emitted in all directions of the 3-dimensional space. [wikipedia]
2.2.1.6 **STOKES' HYPOTHESIS**

Although the factor $\lambda$ is only of importance for compressible fluids, its value is determined in the motion equation with help of Stokes’ hypothesis. It states the following relationship between the two material variables:

$$3\lambda + 2\mu = 0 \text{ oder } \lambda = -\frac{2}{3}\mu$$

Eq. 2-125

The number of material variables is thus reduced from two to one. If this value for $\lambda$ is introduced in Eq. 2-111 and Eq. 2-122, the normal components of the stress tensor can be written as:

$$\sigma_x = -p - \frac{2}{3}\mu \text{div} \vec{v} + 2\mu \frac{\partial u}{\partial x}$$

$$\sigma_y = -p - \frac{2}{3}\mu \text{div} \vec{v} + 2\mu \frac{\partial v}{\partial y}$$

$$\sigma_z = -p - \frac{2}{3}\mu \text{div} \vec{v} + 2\mu \frac{\partial w}{\partial z}$$

Eq. 2-126

The components with mixed indices from Eq. 2-123 remain unchanged.

2.2.1.7 **NAVIER-STOKES EQUATIONS**

Substitution of Eq. 2-122 and Eq. 2-123 and Stokes’ hypothesis Eq. 2-125 into the momentum equation Eq. 2-112 yields the following motion equations:

$$\rho \frac{Du}{Dt} = F_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left[ \mu \left( 2 \frac{\partial u}{\partial x} - \frac{2}{3} \text{div} \vec{v} \right) \right] + \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \frac{\partial}{\partial z} \left[ \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right]$$

$$\rho \frac{Dv}{Dt} = F_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y} \left[ \mu \left( 2 \frac{\partial v}{\partial y} - \frac{2}{3} \text{div} \vec{v} \right) \right] + \frac{\partial}{\partial x} \left[ \mu \left( \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \right) \right] + \frac{\partial}{\partial z} \left[ \mu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right]$$

$$\rho \frac{Dw}{Dt} = F_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z} \left[ \mu \left( 2 \frac{\partial w}{\partial z} - \frac{2}{3} \text{div} \vec{v} \right) \right] + \frac{\partial}{\partial x} \left[ \mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] + \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial w}{\partial y} + \frac{\partial u}{\partial z} \right) \right]$$

Eq. 2-127

with

$$\frac{Du}{Dt} = \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)$$

$$\text{div} \vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$
These differential equations are called Navier-Stokes equations. They were first derived by M. Navier (1827) and S.D. Poisson (1831). The form for an arbitrary coordinate system is:

\[
\rho \frac{D\vec{v}}{Dt} = \nabla p + \text{Div } \tau \tag{2-128}
\]

with

\[
\tau = \mu \left( 2\varepsilon - \frac{2}{3}\delta \text{ div } \vec{v} \right) \tag{2-129}
\]

\[
\delta = \text{ Kronecker unit vector, with:}
\]

\[
\delta_{ij} = \begin{cases} 1 & \text{für } i = j \\ 0 & \text{für } i \neq j \end{cases}
\]

The Navier-Stokes equations are not physically exact because of the assumptions we made in order to make a connection between the stress tensor and the deformation velocity tensor. They would principally have to be proved experimentally.

If we use the assumption of an incompressible fluid again, the Navier-Stokes equation can be further reduced:

(for the case of the x-component)

\[
\rho \frac{Du}{Dt} = F_x - \frac{dp}{dx} + \mu \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 u}{\partial x \partial z} \right] \tag{2-130}
\]

\[
= F_x - \frac{dp}{dx} + \mu \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 u}{\partial x \partial z} \right] \tag{div \ v = 0}
\]

\[
= F_x - \frac{dp}{dx} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)
\]

\[
= F_x - \frac{dp}{dx} + \mu (\Delta u)
\]

This can be done analogously for the y and z-components.
The continuity equation and the Navier-Stokes equations for incompressible fluids are thus:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = F_x - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)
\]

\[
\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = F_y - \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)
\]

\[
\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = F_z - \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)
\]

or in tensor form:

\[
\frac{\partial u_i}{\partial x_i} = 0
\]

\[
\rho \frac{Du_i}{Dt} = F_i - \frac{\partial p}{\partial x_i} + \mu \Delta u_i
\]

We now want to have a closer look at the volume forces \( F_i \). The following forces are important in flowing waters: the gravitational force (we do not assume local variations in the gravitational acceleration), the tidal forces and the Coriolis force.

If only the gravitational force and the Coriolis force are accounted for, the (sum-)vector of volume forces can be written as:

\[
\vec{F} = \begin{pmatrix}
2\rho \omega v \sin \theta \\
-2\rho \omega u \sin \theta \\
-\rho g
\end{pmatrix}
\]

Eq. 2-133

with \( |\omega| = \frac{2 \pi}{86164 \text{s}} \) angular velocity due to rotation of the earth,

\( \theta = \) latitude,

\( g = 9.81 \text{ m/s}^2 \) gravitational acceleration.
2.2.1.2 TURBULENT FLOWS

We generally differentiate between a laminar and a turbulent flow state. If the flow velocity is very small, the flow will be laminar, and if the flow velocity exceeds a certain boundary value, the flow becomes turbulent. Figure 2-14 shows this transition from a “well-ordered” laminar state to a seemingly stochastic and “chaotic” turbulent state for pipe flow. This experiment has already been done by Reynolds around 1860. Reynolds had then shown that the transition from laminar to turbulent can be described by the dimension-less Reynolds-number

\[ \text{Re} = \frac{u \cdot L}{\nu} \quad \text{Eq. 2-134} \]

with \( u \) being the “typical” velocity, \( L \) being the „typical“ characteristic length and \( \nu \) being the kinematical viscosity. The characteristic length \( L \) is used for the description of fluid-transport processes (hydrodynamics, mass transport, heat transport, etc.) in so-called dimensionless index numbers, such as the Reynolds-number or Prandtl-number. It has the dimension of a length, but describes the three-dimensional geometry of a reference system. In simplified terms we apply for the characteristic length in the Reynolds number the diameter if there is a pipe flow or the water depth if we have an open channel flow.

The critical Reynolds number for pipe flow is about 580, so that for example the flow within a pipe with the diameter \( D=0,20 \text{ m} \) is turbulent if the flow velocity is greater than \( \sim 3 \text{ mm/s} \). This observation makes clear that most of the flows with technical relevance are turbulent. One of the few exceptions is for example human blood vessels that have laminar flow as a general rule (…maybe except for the aorta).

Figure 2-14: Reynold’s experiment showing the transition from laminar to turbulent pipe flow, taken from [Van Dyke, 1982].
A turbulent flow can also be characterized by the following properties, according to [Ferziger et al., 1999]:

- The process of turbulence is highly unsteady, so that e.g. the flow velocity at a given point is subject to great variance over time.
- Turbulence is a three-dimensional phenomenon.
- Vortices are an essential part of flow, and the interaction of vortices and the so-called vortex stretching are basic mechanisms that increase and widen turbulence.
- The mixing processes due to common diffusion are often multiplied by turbulence.
- The contact between fluid balls with small and large motion moment also increases because of the strong mixing processes. The acting viscous forces then lead to a loss of kinetic energy, while the loss is translated into inner energy, i.e. heat energy is produced. This process is thus irreversible and dissipative.
- Newer examinations confirm the occurrence of so-called coherent structures. These are reproducible and deterministic processes, which have a big influence on the mixing processes.

This list is very compressed, but it should make clear how complex turbulence is. Many mechanisms of the system are still unsolved, and the rather philosophical question comes up, if we will ever be able to understand the phenomenon turbulence completely and in detail. However, this is not necessary for many questions in engineering, since we are often only interested in average values. Additionally, we are also satisfied with solutions and approximations that are only valid within certain limits, as long as they describe our concrete problem in sufficient detail.

Flows in open natural watercourses are always turbulent. There are two approaches of turbulence, the deterministic and statistical approach:

Figure 2-15: Approaches of the turbulence description (according to Nikora, 2008)
Before we make the transition to numerical calculation approaches of turbulent flow, the terms vortex, vortex stretching and energy cascade shall be illustrated for better understanding of turbulence [Bradshaw, 1999].

A simplified model that helps to understand a turbulent flow assumes the flow to be a superimposition of three-dimensional and locally unstable vortices unto a main flow. These vortices differ in geometric shape and size and are in direct interaction. When considering the coherent structures the vortex is, regardless of whether vertically or horizontally, or the turbulent flow, characterised by geometric shapes and decay processes. Coherent structures are three-dimensional flow zones in which at least one flow variable is receiving some significant correlation with itself or another variable in space and time. This means, that always one vortex is clearly visible in its rotational axis and the main direction of motion. On coherent structures, as shown in Figure 2-15 on the “burst” phenomenon, however, should not be addressed further.

We are therefore considering the static point of view and the energy cascade concept. On the energy cascade initially one vertex is assumed, but which will collapse by vortex stretching into small vortexes [Bradshaw, 1999]. Vortex stretching is a deformation of the vortices making them smaller and adding kinetic energy to them. This kinetic energy is taken from the main flow or bigger vortices, depending on where the work is done. In the smallest vortices, the kinetic energy is transformed into heat energy (Figure 2-15, at the right). Fundamental examinations have shown in the last few years that this energy transport or better this energy cascade, beginning with the main flow over bigger vortices up to the smallest vortices, is no “one-way street”. Consequently, it can happen that the bigger vortices gain energy that comes from smaller vortices. In order to be able to estimate the importance of bigger and smaller vortices, we should have the following two aspects in mind:

- The energy losses of the main flow (kinetic energy is taken) are caused by big vortices to a degree of about 90%.
- Energy dissipation only occurs in the smaller vortices.

It can be seen why the big vortexes are of importance for example for the calculation of flow losses in rivers. They make up the main part of the flow losses caused by turbulence – we also speak of turbulent shear stresses that have an effect here.
2.2.1.2.1 APPROACHES OF NUMERICAL CALCULATIONS OF TURBULENT FLOWS

Turbulent flows are characterized by four main features: diffusion, dissipation, three-dimensionality and length scales. For the numerical calculation of turbulent flows, an averaging of the Navier-Stokes equations of motion is carried out. The averaging can be done with different dimensions (compare Figure 2-16):

- Averaging over time (assuming a statistically stationary flow) or averaging over a measurement (assuming constant boundary)
- Averaging over a space direction, in which the average flow does not vary

The following three approaches are applied most frequently in numerical calculation of turbulent flows:

(1) The most frequent, but also most simple averaging is with respect to time. However, this approach is based on the assumption that the turbulent velocity fluctuations are distributed stochastically, meaning there is a constant mean flow. This averaging leads to the so-called Reynolds Averaged Navier-Stokes equations (short RANS). Additional terms with new variables occur in these partial differential equations because of the averaging. Consequently there are suddenly more variables than equations. In order to close the motion equation system, additional model equations or approximations have to be made, which express the variables as a function of the velocity field of the time-averaged flow. Setting up these model equations is a further research area with the name “turbulence modelling”.

At the time averaging it can be differed between eddy-viscosity models and Reynolds-stress models. In the former case the turbulence is expressed by introducing an additional viscosity, the eddy viscosity. In Reynolds-stress models, the turbulence is considered through direct approaches to individual turbulent stress terms.

(2) The Large Eddy Simulation (LES) does not model the big-scale turbulent structures anymore, but considers them in the direct solution of the NS equations. The influence of small-scale turbulent structures is calculated with help of turbulence models just as for the RANS approach. The main problem with LES is the determination of the limits for the size of vortices – which are big and which are small. The space- and time-discrete systems depend on it.

(3) The Direct Numerical Simulation (DNS, not DANS!!) is the newest and most elaborate numerical approach. Turbulence modelling is not used here, but the NS equations are solved directly here. This requires a high space and time resolution of the simulation, what makes the calculation time-consuming and costly. Even large-scale parallel computer architectures can only calculate flows with Reynolds numbers up to 200 today [Ferziger, 1999]. This is the absolute minimum for engineering and technical problems. There will not be a great change for the applicability of the DNS in the near future, so that it will most likely be limited to fundamental research issues.
The direct numerical simulation (DNS) without any model assumptions is very extensive, so usually the equations are resolved by the use of space or time averaged sizes. The spatial averaging (Large-Eddy Simulation = LES) requires because of the length scales very fine grid resolutions. The momentum transport smaller vortices is usually described by a simple algebraic relationship. Temporal averaging methods (Reynolds models = RANS) dissolve only the average flow and describe the impact of the turbulent flow through empirical approaches.

The approaches LES and DNS will not be discussed in detail here, so the interested reader may refer to the respective technical literature. Only the RANS approach (1) will be further depicted. In the next chapter the time-averaged NS equations will be derived for this purpose.

---

**Figure 2-16: Basic equation and outline of the averaging in space and time**

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2.2.1.2.2 REYNOLDS AVERAGED NAVIER-STOKES EQUATIONS

By hand of a time-averaging of the NS equations and the continuity equation for incompressible fluids, the basic equations for the averaged turbulent flow will be derived in the following. The flow field can then be described only with help of the mean values.

In order to be able to take a time-average, the momentary value is decomposed into the parts mean value and fluctuating value. This is shown graphically in Figure 2-17.

![Figure 2-17: Turbulent velocity fluctuation in pipe flow as a function of time, taken from [Fredsøe, 1990].](image)

The momentary velocity components is \( u \), the time-averaged value is named \( \bar{u} \) and the fluctuating velocity has the letter \( u' \). With help of this definition the decomposition can mathematically be written as:

\[
\begin{align*}
    u &= \bar{u} + u' , \\
    v &= \bar{v} + v' , \\
    w &= \bar{w} + w' , \\
    p &= \bar{p} + p' .
\end{align*}
\]

Eq. 2-135

Analogously for the density and the temperature:

\[
\begin{align*}
    \rho &= \bar{\rho} + \rho' , \\
    T &= \bar{T} + T' ,
\end{align*}
\]

Eq. 2-136

which will however be considered constant in the following.

The chosen averaging method takes the mean values at a fix place in space and averaged over a time span that is large enough for the mean values to be independent of it.

\[
\bar{u} = \frac{1}{\Delta t} \int_{t_i}^{t_i+\Delta t} u \, dt
\]

Eq. 2-137

The time-averaged values of the fluctuating values are defined to be zero:

\[
\bar{u'} = 0, \quad \bar{v'} = 0, \quad \bar{w'} = 0, \quad \bar{p'} = 0
\]

Eq. 2-138
Firstly the continuity equation is averaged. If we substitute the expressions for the velocities from Eq. 2-135 into the continuity equation (see Eq. 2-131) we get:

\[
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} + \frac{\partial \bar{v}'}{\partial y} + \frac{\partial \bar{w}'}{\partial z} = 0 \quad \text{Eq. 2-139}
\]

The time-average of the last equation is written as:

\[
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} + \frac{\partial \bar{v}'}{\partial y} + \frac{\partial \bar{w}'}{\partial z} = 0 \quad \text{Eq. 2-140}
\]

Before we look at the transformation and reduction of Eq. 2-140, a summary of rules for time-averaging shall be given:

\[
\bar{\int \int} = \int \int \bar{\partial} = \partial, \quad \bar{\int \int} = \int \int \bar{\partial} = \partial
\]

\[
\bar{\int \int} = \int \int \bar{\partial} = \partial
\]

\[
\bar{f} = \bar{\int \int f}, \quad \bar{f} + \bar{g} = \bar{\int \int f + g}, \quad \bar{f} \cdot \bar{g} = \bar{\int \int f \cdot g}, \quad \frac{\partial \bar{f}}{\partial s} = \frac{\partial \bar{f}}{\partial s}, \int \int \bar{f} \, ds = \bar{\int \int} f \, ds \quad \text{Eq. 2-142}
\]

but \( \bar{f} \cdot \bar{g} \neq \bar{\int \int f} \cdot \bar{g} \)

The averaged derivatives of the fluctuations are also zero according to these rules, so that the time-averaged continuity equation is:

\[
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \quad \text{Eq. 2-143}
\]

Now the NS equations will be time-averaged. The averaging will be exemplified for the x-component. Beforehand a small transformation of the advection term from Eq. 2-131:

\[
u \frac{\partial \bar{u}}{\partial x} + \nu \frac{\partial \bar{v}}{\partial y} + \nu \frac{\partial \bar{w}}{\partial z} = \frac{\partial (\bar{u}^2)}{\partial x} + \frac{\partial (\bar{uv})}{\partial y} + \frac{\partial (\bar{uw})}{\partial z} - u \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} \right) = \frac{\partial (\bar{u}^2)}{\partial x} + \frac{\partial (\bar{uv})}{\partial y} + \frac{\partial (\bar{uw})}{\partial z} \quad \text{Eq. 2-144}
\]
The expressions for the decomposition of the velocities from Eq. 2-135 are now substituted into the transformed Navier-Stokes equation (see Eq. 2-131) and a time-average is done:

\[
\rho \left\{ \frac{\partial (\bar{u} + u')}{\partial t} + \frac{\partial (\bar{u} + u')^2}{\partial x} + \frac{\partial (\bar{u} + u')(\bar{v} + v')}{\partial y} + \frac{\partial (\bar{u} + u')(\bar{w} + w')}{\partial z} \right\}
\]

\[
= F_x - \frac{\partial (\bar{p} + p')}{\partial x} + \mu \left( \frac{\partial^2 (\bar{u} + u')}{\partial x^2} + \frac{\partial^2 (\bar{u} + u')}{\partial y^2} + \frac{\partial^2 (\bar{u} + u')}{\partial z^2} \right)
\]

Eq. 2-145

Application of the rules from Eq. 2-141 and Eq. 2-142 shows that among others the terms \( \frac{\partial (\bar{u}, u')}{\partial x_j} \frac{\partial u_i}{\partial t} \frac{\partial u_j}{\partial x_j} \) from the equation above can be reduced and the equation can be transformed to:

\[
\rho \left( \frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u} \bar{u}}{\partial x} + \frac{\partial \bar{u} u}{\partial y} + \frac{\partial \bar{u} v}{\partial y} + \frac{\partial \bar{u} w}{\partial z} + \frac{\partial \bar{u} w}{\partial z} \right)
\]

\[
= F_x - \frac{\partial \bar{p}}{\partial x} + \mu \left( \frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2} \right)
\]

Eq. 2-146

Further small transformations, for example a repeated application of the product rule and the continuity equation to the advection term, lead to a form of the time-averaged NS equations for all three directions as:

\[
\rho \left( \frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u} \bar{u}}{\partial x} + \frac{\partial \bar{v} \bar{u}}{\partial y} + \frac{\partial \bar{w} \bar{u}}{\partial z} \right) = F_x - \frac{\partial \bar{p}}{\partial x} + \mu \Delta \bar{u} - \rho \left( \frac{\partial \bar{u} \bar{u}}{\partial x} + \frac{\partial \bar{u} \bar{v}}{\partial y} + \frac{\partial \bar{u} \bar{w}}{\partial z} \right)
\]

Eq. 2-147

\[
\rho \left( \frac{\partial \bar{v}}{\partial t} + \frac{\partial \bar{u} \bar{v}}{\partial x} + \frac{\partial \bar{v} \bar{v}}{\partial y} + \frac{\partial \bar{w} \bar{v}}{\partial z} \right) = F_y - \frac{\partial \bar{p}}{\partial y} + \mu \Delta \bar{v} - \rho \left( \frac{\partial \bar{u} \bar{v}}{\partial x} + \frac{\partial \bar{v} \bar{v}}{\partial y} + \frac{\partial \bar{v} \bar{w}}{\partial z} \right)
\]

Eq. 2-148

Or in tensor form:

\[
\rho \frac{D \bar{u}_i}{Dt} = F_i - \frac{\partial \bar{p}}{\partial x_i} + \mu \Delta \bar{u}_i - \rho \left( \frac{\partial \bar{u}'_i \bar{u}'_j}{\partial x_j} \right)
\]

Eq. 2-149

\( \text{Reynolds' stress} \)
From now on the time-averaged fields will not be overlined anymore. So for example \( u \) stands for the time-averaged velocity component in direction of the x-axis.

We pay attention to the last two terms of the right side of Eq. 2-148:

\[
\mu \Delta u_i - \rho \left( \frac{\partial u_i}{\partial x_j} \right)
\]

\[
= \mu \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} \right) - \rho \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} \right) - \rho \frac{\partial u_i}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} \right) \tag{Eq. 2-149}
\]

\[
= \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_i}{\partial x_j} \right) - \rho \frac{\partial u_i}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} \right)
\]

The expression in the brackets above corresponds to the total shear stress:

\[
\tau_{ij} = \mu \frac{\partial u_i}{\partial x_j} - \rho \frac{\partial u_i}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} \right) \tag{Eq. 2-150}
\]

If we compare to the Navier-Stokes equations Eq. 2-131, it is conspicuous that besides the viscous part an additional term has been added to the total shear stress. This term results from the time-average and is generally the dominant part of the total shear stress. Since the term only appears due to the Reynolds average, it is called Reynolds stress or apparent turbulent shear stress. As stated in the introduction to the RANS approach, to lead to the closure of the equation system, an approximation for the Reynolds stresses has to be done, which sets in relation the apparent shear stresses with the velocity field of the average flow.

With the approach of the eddy viscosity principle after Boussinesq 1877, the general time-averaged NS equations, also called Reynolds equations, can thus be written in tensor form as:

\[
\rho \left( \frac{Du_i}{Dt} \right) = F_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \tau_{ij} \tag{Eq. 2-151}
\]

\[
\text{with } \tau_{ij} = \mu \frac{\partial u_i}{\partial x_j} + \rho \left( \nu_T \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \right)
\]
2.2.1.2.2.1 Apparent shear stress

In contrast to the bed shear stress, the turbulent shear stress or shear strain considers amongst others a phenomenon called **apparent shear stress**, which occurs between water layers of different speed in the water column. If layers with different speeds are streaming next to each other, it comes about shear stress or apparent shear stress through an exchange of momentum. One important assumption implies that viscosity influences due to terms of adhesion only become noticeable in a range near the wall, provided that this influences are not covered by the influence of a coarse roughness.

If \( u \) is the flow velocity and \( z \) is the coordinate vertical to the flow, the difference in the flow velocity between two layers, which are apart from each other with the distance \( \ell \), can be described in a first approximation by the following equation:

\[
\Delta u = \ell \cdot \frac{d\bar{u}}{dz}
\]

Eq. 2-152

with:
- \( \ell \) = distance between two layers in [m]
- \( \bar{u} \) = average flow velocity in [m/s]

A fluid element which is initially located at \( z \), has at the location \((z + \ell)\) a smaller velocity than his new environment in which it is carried. This velocity difference is a measure of the fluctuation velocity \( \Delta u \) in \( x \) direction. For the turbulent mass exchange Prandtl assumes, that the exchange rate has the same scale as the velocity difference between the two layers which are apart from each other in the distance of \( \ell \). This is due to the fact, that the liquid bale collide with a velocity of this scale.

2.2.1.2.2.2 Approach of Boussinesq

A relatively old approach to this is the principle of eddy viscosity, which in 1877 was formulated by Boussinesq and is still the basis of many practical turbulence models [Rodi, 1993]. The eddy viscosity principle considers for the turbulent apparent shear stress analogous to the viscous shear stress in laminar flow, that there is a proportionality to the velocity gradients of the mean flow. This can be expressed as:

\[
-\bar{u}_i' \bar{u}_j' = \nu_T \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij}
\]

Eq. 2-153
The turbulent kinetic energy $k$ is defined by:

$$k = \frac{1}{2} (u^2 + v^2 + w^2)$$  \hspace{1cm} \text{Eq. 2-154}

The so-called turbulent eddy viscosity $\nu_t$ is a proportionality factor $\nu_t$ (no physical property!). It depends intensively on the degree of turbulence. That means $\nu_t$ varies within the fluid flow and, depending on the flow condition. The second term of the Eq. 2-153, the Kronecker delta $\delta_{ij}$ (see Eq. 2-127), ensures that the equation is also valid for the normal tension, whose sum is according to Eq. 2-154 2$k$.

The principle of the eddy viscosity is not a modelling of turbulence in the true sense, it is only the basic framework. Approximations are considered as turbulence models, if they provide an approach to calculate the eddy viscosity $\nu_t$.

2.2.1.2.3 APPLICATIONS AND APPROACHES FOR TURBULENCE MODELLING

Turbulences describe spatial and temporal variation in the mean flow field with seemingly unsystematic character. They are caused by local shear strains of the velocity field and to a lesser extent by pressure fluctuations due to surface waves. The turbulence is no movement or material property, but is dependent on their environment. When turbulence flow energy is converted into heat. Thus, large vortices dispense in a vortex cascade energy to smaller vortices, as the increase in the kinetic energy at the molecular level leads to a temperature increase.

2.2.1.2.3.1 Closing models of zeroth, first and second order

The names of zero-, one- and two-equation models relate to the way the turbulent stresses are expressed or how many additional conservation equations are needed. With zero-equation models, following the approach of Boussinesq (1877) we assume that the turbulent stresses are proportional to the flow velocity. In one-equation models additional partial differential equations for the velocity scale are used for turbulences. Another partial differential equation for the length scale is added for the two-equation models. This group also includes the well-known $k-\varepsilon$ and $k-\omega$ models. Stress models need for all components of the stress tensor all partial differential equations.

Approaches to determine the turbulent eddy viscosity provide the described closure models zeroth, first and second order. As the conditions for the Rouse-profile are a logarithmic velocity distribution and a parabolic distribution of the turbulent viscosity, the mixing length model after Prandtl (1925) may be recognized for an approach of the first-order.

The following other approaches can be selected for the eddy viscosity:

- a constant value $\nu_t$ (constant eddy viscosity)
- a time-variable function of the local gradient of the flow velocities (Smagorinsky)
- an one-dimensional $k$-model to solve an additional equation for the transport of a turbulent kinetic energy
- a two-dimensional $k-\varepsilon$ model for the solution of two additional transport equations
- a mixed $k-\varepsilon$ model for the solution of two additional transport equations in the vertical after Rodi and Smagorinsky`'s approach in the horizontal.
2.2.1.2.3.1.1 Approach for a constant eddy viscosity

For the description of the turbulent velocity fluctuation an approach to the eddy viscosity $v_t$ is involved for the diffuse momentum transport. Thereby, the eddy viscosity $v_t$ is in contrast to the kinematic viscosity not a physical property, but depending on the current flow condition, thus a function of time and place. The eddy viscosity is sometimes also exchanged by setting the Peclet-number $Pe$- and thereby takes values between 15 and 40. It may be calculated out of the eddy viscosity $E$:

$$Pe = \frac{v_x \cdot \rho \cdot dx}{E} = \frac{v_x \cdot dx}{v_t}$$

Eq. 2-155

With the introduction of the eddy viscosity, the viscous shear stress of a fluid can be described by the principle of eddy viscosity after Boussinesq 1877 (see 2.2.1.2.2.2).

$$-u^{'}v^{'} = v_t \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - \frac{2}{3} k \cdot \delta_{x,y}$$

Eq. 2-156

2.2.1.2.3.1.2 Approach for the $k$-$\varepsilon$ model

The $k$-$\varepsilon$ model is a matter of a second-order closure model, which consults the turbulent kinetic energy $k$ and energy dissipation $\varepsilon$ for the determination of $v_t$. This requires that two additional equations have to be solved for determining $k$ and $\varepsilon$. Considering the energy dissipation $\varepsilon$ as a representation by means of a dimensional analysis it shows the connection to the characteristic length $\ell$:

$$\varepsilon = c_D \frac{k^{3/2}}{\ell}$$

Eq. 2-157

with:
- $c_D = \text{coefficient of drag in [-]}$
- $\ell = \text{mixing length in [m]}$
- $k = \text{term of diffusion}$
- $\varepsilon = \text{term of dissipation}$

If it is combined with the Prandtl-Kolmogorov equation, you get the following expression for the coefficient of eddy viscosity:

$$v_t = c_\mu \frac{k^2}{\varepsilon}$$

Eq. 2-158

with:
- $c_\mu = \text{coefficient} = 0.09 \text{ in [m}^2\text{/s]}$
- $0$ für $G < 0$
- $0.09$ für $G > 0$
- $k = \text{term of diffusion}$
- $\varepsilon = \text{term of dissipation}$
The $k$-$\varepsilon$ model takes into account the impact of the layered flow on the mixture and the turbulent viscosity by a buoyancy term:

$$G = \frac{V_i \cdot g \cdot \frac{\partial \rho}{\partial \varepsilon}}{\sigma_i \cdot \rho} \quad \text{Eq. 2-159}$$

For stable stratification $G$ is negative. For unstable stratification $G$ is positive. If there is no stratification of the flow, $G$ is zero. The following equations result from the $k$-$\varepsilon$ model:

$$\frac{\partial k}{\partial t} + \bar{u} \ \text{grad} \ k = \text{div} \ \frac{V_i}{\sigma_k} \cdot \text{grad} \ k + \frac{P}{\rho} \ \text{grad} \ \bar{u} + G - \varepsilon \quad \text{Eq. 2-160}$$

$$\frac{\partial \varepsilon}{\partial t} + \bar{u} \ \text{grad} \ \varepsilon = \text{div} \ \frac{V_i}{\sigma_\varepsilon} \cdot \text{grad} \ \varepsilon + c_i \varepsilon \left( \frac{P}{\rho} \ \text{grad} \ \bar{u} + (1 - c_3 \varepsilon) G \right) - c_2 \varepsilon \frac{\varepsilon^2}{k} \quad \text{Eq. 2-161}$$

For the two additional equations also more boundary conditions for the solution are now necessary. At the river bed, it is assumed that the boundary of the modelled area lies outside of the viscous sub layer in the distance $y'$, since for the viscous sub layer no solution is known. This leads to boundary conditions at the river bed to:

$$\varepsilon_{Sohle} = \frac{u_h^3}{k \cdot y'} \quad \text{and} \quad k_{Sohle} = \frac{u_h^2}{\sqrt{c_i \mu}} \quad \text{with} \ u_h = \frac{\tau_h}{\sqrt{\rho}} \quad \text{Eq. 2-162}$$

For the boundary condition at the free surface it can be assumed that there is no exchange of turbulent kinetic energy with the atmosphere. Also it is assumed that no shear stresses between air and water occur, whose velocities in amount and direction are the same. This assumption without the influence of wind is described on the Neumanns boundary condition:

$$\frac{\partial k_{wsp}}{\partial n_s} = 0 \quad \text{Eq. 2-163}$$

For the turbulent energy dissipation after Celik & Rodi (1984), the Dirichlets boundary condition applies with:

$$\varepsilon_{wsp} = \frac{k_{wsp}^{3/2}}{0.18 h} \quad \text{Eq. 2-164}$$
2.2.1.2.3.1.3 Approach for the mixed $k$-$\varepsilon$ model

The mixed-$k$-$\varepsilon$ model for the eddy viscosity solves two additional transport equations after Rodi in the vertical:

\[
\frac{\partial k}{\partial t} = \frac{\partial}{\partial z} \left( \frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial z} \right) + P + G - \varepsilon \tag{Eq. 2-165}
\]

\[
\frac{\partial \varepsilon}{\partial t} = \frac{\partial}{\partial z} \left( \frac{\nu_t}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial z} \right) + c_{i\varepsilon} \varepsilon \left( \frac{P + c_{3\varepsilon} G}{k} \right) + c_{2\varepsilon} \frac{\varepsilon^2}{k} \tag{Eq. 2-166}
\]

Therefore the coefficients of $P$ and $G$ are:

\[
P = \nu \left( \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 \right) \quad \text{and} \quad G = \frac{g}{\rho} \cdot \frac{v}{\sigma_\varepsilon} \left( \frac{\partial p}{\partial z} \right) \tag{Eq. 2-167}
\]

If $G$ is less than zero then one speaks of a stable stratification and the parameter $c_{3\varepsilon}$ becomes zero. If the stratification is unstable, the coefficient $G$ takes a value greater than zero and the parameters $c_{3\varepsilon}$ does not behave constant. Typically, he nevertheless is applied consistently.

The empirical parameters for the mixed $k$-$\varepsilon$ model are chosen to:

- $c_{3\varepsilon}$ = empirical constant as a result to stratification in [-]
- $c_{3\varepsilon}$ = 0 unstable stratification
- $c_{3\varepsilon}$ = 1 stable stratification
- $c_{i\varepsilon}$ = 1.44 = empirical constant in [-]
- $c_{2\varepsilon}$ = 1.92 = empirical constant in [-]
- $\sigma_k$ = 1.0 = empirical constant in [-]
- $\sigma_\varepsilon$ = 1.3 = empirical constant in [-]

For the various mixed $k$-$\varepsilon$ model also the Prandtl-number is modified:

\[
\sigma_t = \sqrt{(1 + 10 \cdot Ri)^{-1}} \left( 1 + 10 \cdot Ri \right) \tag{Eq. 2-168}
\]

The Richardson-number $Ri$ is a dimensionless parameter for turbulence (values between approximately 10 to 0.1). The smaller $Ri$, the more likely are turbulences. The turbulent Richardson-number $Ri$ describes the stratification of the flow to:

\[
Ri = -\frac{g}{\rho} \cdot \frac{\partial p}{\partial z} \left( \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 \right)^{-1} \tag{Eq. 2-169}
\]

Further descriptions are given by Malereck (2001), Schroeder et Forkel (1999) and Rodi (1993).
2.2.1.3 **DEPTH-AVERAGED SHALLOW WATER EQUATIONS**

The Reynolds equations that were derived in the last chapter describe the motion processes of a flow in all three dimensions. This elaborate approach is appropriate for example for the mathematical calculation of flows close to constructs or in strongly meandering rivers, so basically everywhere where the flow is dominated by three-dimensional effects. Since the numerical three-dimensional calculation is still very costly, it makes sense to reduce the Reynolds equations for calculations with simpler flow conditions. The depth-averaged two-dimensional flow equations, also called shallow water equations, provide an example. The shallow water equations are obtained, as the name suggests, by averaging the Reynolds equations over the depth. The following conditions have to be met in order for the shallow water equations to be applicable:

- the vertical momentum exchange is negligible and the vertical velocity component \( w \) is a lot smaller than the horizontal components \( u \) and \( v \):
  \[
  w << u \quad \text{and} \quad w << v.
  \]

- the pressure gain is linear with the depth (parallel flow lines \( \Rightarrow \) hydrostatic pressure distribution):
  \[
  p(z) = \gamma \cdot z
  \]
  with \( z \) being the depth measured from the water surface
  and \( \gamma = \rho \cdot g \).

These assumptions make it possible to reduce the basic system of equations to only three equations: the continuity equation (as usual) and the motion equations in direction of the \( x \)- and \( y \)-axis, as follows:

**x-component:**

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = F_x - \gamma \left( \frac{\partial (z_u + h)}{\partial x} + \left( \frac{\partial}{\partial x} \tau_{xy} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{xz} \right) \right)
\]

Eq. 2-170

**y-component:**

\[
\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = F_y - \gamma \left( \frac{\partial (z_v + h)}{\partial y} + \left( \frac{\partial}{\partial x} \tau_{xy} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{yz} \right) \right)
\]

Eq. 2-171

Prior to integrating the momentum equations over the depth, we will work on the continuity equation, as a small practical example to warm up. An important tool for the depth-integration are the so-called kinematical boundary conditions that give information about the change of the water surface over time.
Kinematical boundary condition at the free surface:
An expression for the surface motion is derived.

\[ z_{\text{Surface}} = \text{Sohle} + \text{Wassertiefe} = z_0 + h \]

\[ w|_{z_0+h} = \frac{D(z_0 + h)}{Dt}, \text{ where } (z_0 + h) = f(t, x, y) \]

\[ \Rightarrow w|_{z_0+h} = \frac{\partial (z_0 + h)}{\partial t} + \frac{\partial x}{\partial t} \frac{\partial (z_0 + h)}{\partial x} + \frac{\partial y}{\partial t} \frac{\partial (z_0 + h)}{\partial y}, \text{ (chain rule)} \]

\[ \Rightarrow w|_{z_0+h} = \frac{\partial z_0}{\partial t} + \frac{\partial h}{\partial t} + u|_{z_0+h} \frac{\partial (z_0 + h)}{\partial x} + v|_{z_0+h} \frac{\partial (z_0 + h)}{\partial y} \]

\[ \Rightarrow w|_{z_0+h} = \frac{\partial h}{\partial t} + u|_{z_0+h} \frac{\partial (z_0 + h)}{\partial x} + v|_{z_0+h} \frac{\partial (z_0 + h)}{\partial y} \quad \text{Eq. 2-172} \]

Kinematical boundary condition at the river bed:
Ground is impermeable \(\Rightarrow\) no mass flux perpendicular to bed.

\[ \vec{u} \cdot \vec{n}_o = 0, \quad \vec{n}_o = \text{normal vector of the river bed} \]

\[ \Rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \frac{\partial z_0}{\partial x} \\ \frac{\partial z_0}{\partial y} \\ -1 \end{pmatrix} \quad \text{Eq. 2-173} \]

\[ \Rightarrow w|_{z_0} = u|_{z_0} \frac{\partial z_0}{\partial x} + v|_{z_0} \frac{\partial z_0}{\partial y} \]
Other tools are the Leibniz theorem and the fundamental theorem of integration, that both shall be restated in this place (just to make sure):

Leibniz – theorem:

\[
\frac{\partial}{\partial x} \int_{z_0}^{z_0+h} u \, dz = \int_{z_0}^{z_0+h} \frac{\partial u}{\partial x} \, dz + u\bigg|_{z_0+x+h} \frac{\partial (z_0+h)}{\partial x} - u\bigg|_{z_0} \frac{\partial z_0}{\partial x}
\]

Eq. 2-174

Integration theorem:

\[
\int_{z_0}^{z_0+h} \left( \frac{\partial u}{\partial z} \right) \, dz = u\bigg|_{z_0+x+h} - u\bigg|_{z_0}
\]

With these instruments we tackle the integration of the continuity equation about the vertical axis, that means between the bed \(z_0\) and the free surface \(z_0+h\) (with \(h\) being the water depth):

\[
\int_{z_0}^{z_0+h} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \, dz = 0
\]

\[
\int_{z_0}^{z_0+h} \frac{\partial u}{\partial x} \, dz + \int_{z_0}^{z_0+h} \frac{\partial v}{\partial y} \, dz + \int_{z_0}^{z_0+h} \frac{\partial w}{\partial z} \, dz = 0
\]

\[
\frac{\partial}{\partial x} \int_{z_0}^{z_0+h} u \, dz - u\bigg|_{z_0+h} \frac{\partial (z_0+h)}{\partial x} + u\bigg|_{z_0} \frac{\partial z_0}{\partial x}
\]

\[
+ \frac{\partial}{\partial y} \int_{z_0}^{z_0+h} v \, dz - v\bigg|_{z_0+h} \frac{\partial (z_0+h)}{\partial y} + v\bigg|_{z_0} \frac{\partial z_0}{\partial y}
\]

\[
+ w\bigg|_{z_0+h} = 0
\]

Eq. 2-175
In a little different form and substituting the kinematical boundary conditions yields:

\[
\frac{\partial}{\partial x} \int_{z_o}^{z_o+h} u \, dz + \frac{\partial}{\partial y} \int_{z_o}^{z_o+h} v \, dz
\]

\[
- u\big|_{z_o+h} \frac{\partial (z_o + h)}{\partial x} - v\big|_{z_o+h} \frac{\partial (z_o + h)}{\partial y} + w\big|_{z_o+h}
\]

\[= \frac{\partial h}{\partial t}\frac{\partial}{\partial t}\frac{\partial h}{\partial t}\] (temporal change of the surface)

\[+ u\big|_{z_o} \frac{\partial z_o}{\partial x} + v\big|_{z_o} \frac{\partial z_o}{\partial y} - w\big|_{z_o} = 0\]

\[= 0\] (no mass flux perpendicular to the river bed)

\[\Rightarrow \frac{\partial}{\partial x} \int_{z_o}^{z_o+h} u \, dz + \frac{\partial}{\partial y} \int_{z_o}^{z_o+h} v \, dz + \frac{\partial h}{\partial t} = 0\]

Eq. 2-176

Introduction of discharge as an integral of the flow velocity over the depth, and the depth-averaged flow velocities \( \bar{u} \) and \( \bar{v} \) (not to be confused with the symbols for the time-averaged fields mentioned before) we get:

\[q_x = \int_{z_o}^{z_o+h} u \, dz = \bar{u}h \text{ und } q_y = \int_{z_o}^{z_o+h} v \, dz = \bar{v}h\]

Eq. 2-177

The depth-integrated continuity equation can thus finally be stated as:

\[\frac{\partial (\bar{v} h)}{\partial x} + \frac{\partial (\bar{v} h)}{\partial y} + \frac{\partial h}{\partial t} = 0\]

Eq. 2-178

The depth-integrated continuity equation shows that the difference between the flow into and out of a volume of water comes with a change of the water depth. The derivation of Eq. 2-178 has been done without simplifying assumptions. The equation represents the conditions exactly.

Consequently we will do the depth-integration of the momentum equations. The procedure is a little lengthy, but we will go through it in detail anyway exemplary for the x-component.

\[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -g \frac{\partial (z_o + h)}{\partial x} + \frac{1}{\rho} \frac{\partial \tau_{xx}}{\partial x} + \frac{1}{\rho} \frac{\partial \tau_{xy}}{\partial y} + \frac{1}{\rho} \frac{\partial \tau_{xz}}{\partial z} + F_x\]

Eq. 2-179
And depth-integrated:

$$\int_{z_o}^{z_{o+h}} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \, dz + \int_{z_o}^{z_{o+h}} \frac{\partial}{\partial x} \left( z_o + h \right) \, dz = \int_{z_o}^{z_{o+h}} \frac{1}{\rho} \left( \frac{\tau_{xx}}{\partial x} + \frac{\tau_{xy}}{\partial y} + \frac{\tau_{xz}}{\partial z} \right) \, dz + \frac{1}{\rho} \int_{z_o}^{z_{o+h}} F_z \, dz$$

Eq. 2-180

The components of this equation can be transformed so that the kinematical boundary condition can be applied:

$$\int_{z_0}^{z_{o+h}} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \, dz = \int_{z_0}^{z_{o+h}} \frac{\partial}{\partial x} \left( z_o + h \right) \, dz + \int_{z_0}^{z_{o+h}} \frac{\partial}{\partial y} \left( z_o + h \right) \, dz + \int_{z_0}^{z_{o+h}} \frac{\partial}{\partial z} \left( z_o + h \right) \, dz$$

Eq. 2-181

For better overview, the depth-integration will be done separately for each term. The first term includes the derivative and the advection part. Then the Leibniz theorem will be applied to both the derivative and the two horizontal advection terms, and the fundamental theorem of integration will be applied to the third advection term:

$$\int_{z_0}^{z_{o+h}} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \, dz = \frac{\partial}{\partial t} \int_{z_0}^{z_{o+h}} u \, dz + \int_{z_0}^{z_{o+h}} \frac{\partial u}{\partial x} \, dz + \int_{z_0}^{z_{o+h}} \frac{\partial u}{\partial y} \, dz + \int_{z_0}^{z_{o+h}} \frac{\partial u}{\partial z} \, dz$$

Eq. 2-182
The terms in the parentheses are zero because of the kinematical boundary conditions and we obtain the following for the first term:

\[
\int_{z_s}^{z_i} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) dz = \frac{\partial (u h)}{\partial t} + \frac{\partial}{\partial x} \int_{z_s}^{z_i} u^2 dz + \frac{\partial}{\partial y} \int_{z_s}^{z_i} u v dz
\]

Eq. 2-183

The second term, in the following referred to as pressure term, will remain unchanged.

The third term, containing the viscous terms, will partly be transformed with the Leibniz theorem and partly with the fundamental theorem of integration. First the horizontal viscous parts:

\[
\int_{z_s}^{z_i} \frac{\partial \tau_{xx}}{\partial x} dz = \frac{\partial}{\partial x} \int_{z_s}^{z_i} \tau_{xx} dz - \tau_{xx} \bigg|_{z_s}^{z_i} \frac{\partial (z_s + h)}{\partial x} + \tau_{xx} \bigg|_{z_s}^{z_i} \frac{\partial z_s}{\partial x}
\]

Eq. 2-184

\[
\int_{z_s}^{z_i} \frac{\partial \tau_{yy}}{\partial y} dz = \frac{\partial}{\partial y} \int_{z_s}^{z_i} \tau_{yy} dz - \tau_{yy} \bigg|_{z_s}^{z_i} \frac{\partial (z_s + h)}{\partial y} + \tau_{yy} \bigg|_{z_s}^{z_i} \frac{\partial z_s}{\partial y}
\]

Eq. 2-185

The vertical viscous term will be integrated with help of the fundamental theorem:

\[
\int_{z_s}^{z_i} \frac{\partial \tau_{zz}}{\partial z} dz = \tau_{zz} \bigg|_{z_s}^{z_i} - \tau_{zz} \bigg|_{z_s}^{z_i}
\]

Eq. 2-185

Depth-integrating the volume forces in direction of the x-axis with reference to Eq. 2-131 yields:

\[
\frac{1}{\rho} \int_{z_s}^{z_i} F_x dz = \frac{1}{\rho} \int_{z_s}^{z_i} (2 p \omega v \sin \theta) dz = h \omega v \sin \theta
\]

Eq. 2-186
Considering equations Eq. 2-181 to Eq. 2-183, we get the preliminary depth-integrated momentum equation in direction of the x-axis:

$$\frac{\partial}{\partial t} (u_h) + \frac{\partial}{\partial x} \left( \int_{z_o}^{z_i} u^2 z h \ dz + \frac{1}{\rho} \frac{\partial}{\partial x} (\rho (\tau_{x \ h})) \right) + \frac{1}{\rho} \frac{\partial}{\partial y} (\rho (\tau_{y \ h})) = - \int_{z_o}^{z_i} \frac{\partial}{\partial x} (u_h z h) \ dz + \frac{1}{\rho} \frac{\partial}{\partial x} (\rho (\tau_{x \ h})) + \frac{1}{\rho} \frac{\partial}{\partial y} (\rho (\tau_{y \ h}))$$

$$+ \frac{\tau_{xx}}{\rho} \bigg|_{z_o + h} - \frac{\tau_{xy}}{\rho} \bigg|_{z_o + h} + \frac{\tau_{xz}}{\rho} \bigg|_{z_o + h} = \tau_{\text{wind}, x}$$

Eq. 2-187

The following expression will be introduced for the wind and bed shear stresses:

$$- \frac{\tau_{xx}}{\rho} \bigg|_{z_o + h} + \frac{\partial}{\partial x} (z_o + h) \frac{\partial}{\partial x} \rho \bigg|_{z_o + h} + \frac{\tau_{xx}}{\rho} \bigg|_{z_o + h} = \tau_{\text{wind}, x}$$

and

Eq. 2-188

$$+ \frac{\tau_{xx}}{\rho} \bigg|_{z_o} - \frac{\tau_{xy}}{\rho} \bigg|_{z_o} + \frac{\tau_{xz}}{\rho} \bigg|_{z_o} = - \tau_{\text{so}, x}$$

In analogy to the time-averaging of the NS equations, a division of the momentary vertical field into a depth-integrated mean value part and a deviation of the mean value is done. For the velocity component in direction of the x-axis this division is:

$$u(z) = \bar{u} + \tilde{u}$$

Eq. 2-189

with $\bar{u}$ being the mean velocity over the vertical axis (not to be confused with the time-averaged mean value that has the same symbol) and $\tilde{u}$ being the deviation of the mean velocity. The following rule of integration holds for this division:

$$\int_{z_o}^{z_i} (\bar{u} + \tilde{u}) (\bar{u} + \tilde{u}) \ dz = \int_{z_o}^{z_i} \bar{u} \ dz + \int_{z_o}^{z_i} \tilde{u} \ dz + 2 \int_{z_o}^{z_i} \bar{u} \tilde{u} \ dz$$

Eq. 2-190

with $\int_{z_o}^{z_i} \tilde{u} \ dz = 0$, $\tilde{u} = u(z) - \bar{u}$
Substituting the expressions for the wind and bed shear stresses as well as the parts of the division from Eq. 2-187 into Eq. 2-185 yields the following depth-averaged momentum equation in direction of the x-axis:

$$\frac{\partial (u h)}{\partial t} + \frac{\partial \left( \frac{u^2 h}{2} \right)}{\partial x} + \frac{\partial (u v h)}{\partial y} + \frac{\partial \int_{z_b}^{z_t} \ddot{u} \ddot{d} z}{\partial x} + \frac{\partial \int_{z_b}^{z_t} \ddot{v} \ddot{d} z}{\partial y} = 0$$

Eq. 2-191

As one of the last steps, the depth-integrated form of the continuity equation is isolated from the first three terms on the left side of the equation with help of partial differentiation.

$$\frac{\partial (\ddot{u} h)}{\partial t} + \frac{\partial \left( \ddot{u}^2 h \right)}{\partial x} + \frac{\partial \ddot{u} \ddot{v} h}{\partial y} = 0$$

Eq. 2-192

Subsequently, we divide by the water depth and group the “fluctuating terms” with the viscous terms. We now get the momentum equation in direction of the x-axis:

$$\frac{\partial \ddot{u}}{\partial t} + \ddot{u} \frac{\partial \ddot{u}}{\partial x} + \ddot{v} \frac{\partial \ddot{u}}{\partial y} = 0$$

Eq. 2-193

For a better overview, the general form of the depth-averaged continuity equation and the depth-averaged momentum equations, the so-called shallow water equations:

$$\frac{\partial h}{\partial t} + \frac{\partial (u h)}{\partial x} = 0$$

Eq. 2-194

where $a_F = \text{“acceleration component“ of the volume forces.}$
The individual terms will now be explained shortly. The first term on the left side is the rate of change (over time) and the second term the convective momentum transport. The right side includes the gravitation force (third term) and the fourth term is the diffuse momentum transport with:

\[
\tau_{i,j,(m+1)} = \mu \frac{\partial u_i}{\partial x_j} + \rho \left( \nu T \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \right)
\]

Eq. 2-195

The fifth term is the dispersive momentum transport, which is a mathematical result similar to the Reynolds stresses – only by depth-integration. For uniform and homogeneous flows this term can be neglected. It describes the exchange processes due to vertical non-uniformities. If there is a strong secondary flow, for example because of strong meandering, the dispersive term becomes important. The sixth term combines the forces from the outside, that is bed shear stress and wind shear stress. The wind shear stress can be neglected most of the times, however.

The bed shear stress is bound to the depth-averaged velocity by a quadratic velocity law.

\[
\tau_o = c_f \rho u^2 \quad \text{(allgemein)}
\]

\[
\tau_o = c_f \rho \bar{u} \left| \bar{u} \right| \quad \text{Eq. 2-196}
\]

Here, \(c_f\) is the friction coefficient, which can be determined according to the flow laws of Gauckler-Manning-Strickler or Darcy-Weisbach.

Darcy-Weisbach’s flow law is definitely preferred.

\[
\tau_o = \frac{\lambda}{8} \rho \bar{u} \left| \bar{u} \right| \quad \Rightarrow \quad c_f = \frac{\lambda}{8}
\]

Eq. 2-197

where drag coefficient \(\lambda\) can be determined with help of Colebrook & White’s formula as a function of the fields:

\[
f = \quad \text{cross section of form drag coefficient},
\]

\[
Re = \quad \text{Reynolds number},
\]

\[
k_s = \quad \text{equivalent sand roughness},
\]

\[
r_hy = \quad \text{hydraulic radius}.
\]

When using the Gauckler-Manning-Strickler flow law, it is of importance that the empiric \(k_S\)-value depends on \(Re\) and \(r_{hy}\), i.e. the discharge state. The \(k_S\)-value is thus no parameter of general, global validity, but it has to be adapted to the discharge state.
If we look at the derived shallow water equations Eq. 2-192, it is conspicuous that we have a closure problem again. In order to solve the problem, the same procedure as for the time-averaging can be applied. We simplify and summarize the fourth and fifth terms of Eq. 2-192 to:

\[
\tau_{g,(m+1)} - \rho \left( \overline{\mathbf{u}} \cdot \frac{\partial \mathbf{u}}{\partial x_i} \right) = \tau_{g,g} = \tau_g
\]

\[
\nu_g = \frac{\nu_{g,0}}{\omega} + \nu_T + \nu_D \quad \text{Eq. 2-198}
\]

\[
\tau_g = \rho \nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

We see that the total viscosity is composed of a molecular part, a turbulent part and a dispersive part.

Further transformations yield:

\[
\tau_{ij} = \rho \nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

\[
\varepsilon_{ij} = \rho \nu_{\theta,ij}
\]

\[
\Rightarrow \tau_{ij} = \varepsilon_{ij} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

These simplifications make it possible to write the shallow water equations in the following form:

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u} = -g \left( \frac{\partial}{\partial x} \left( z_0 + h \right) + \frac{1}{h \rho} \frac{\partial}{\partial x} \left( h \tau_{x,x} \right) + \frac{1}{h \rho} \frac{\partial}{\partial y} \left( h \tau_{x,y} \right) \right)
\]

\[
- \frac{1}{h} \tau_{x,0,x} - \frac{1}{h} \tau_{\text{wind},x,x} + 2 \omega \nu \sin \theta
\]

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{v} + \nu \nabla^2 \mathbf{v} = -g \left( \frac{\partial}{\partial y} \left( z_0 + h \right) + \frac{1}{h \rho} \frac{\partial}{\partial x} \left( h \tau_{y,y} \right) + \frac{1}{h \rho} \frac{\partial}{\partial y} \left( h \tau_{y,y} \right) \right)
\]

\[
- \frac{1}{h} \tau_{y,0,y} - \frac{1}{h} \tau_{\text{wind},y,y} - 2 \omega u \sin \theta
\]

And in tensor form:

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -g \left( \frac{\partial}{\partial x} \left( z_0 + h \right) + \frac{1}{h \rho} \frac{\partial}{\partial x} \left( h \tau_{x,x} \right) \right) - \frac{1}{h} \tau_{x,0,x} - \frac{1}{h} \tau_{\text{wind},x,x} + a_T
\]

\[
\text{Eq. 2-201}
\]

In this equation we have mixed the dispersion part with the diffusion, so to say, by introducing a total viscosity. However, this way does not provide a pure turbulence modelling anymore, independent from the turbulence model used, since this integrated approach also models the dispersion.
In Eq. 2-198 and Eq. 2-199 all fields are time- and depth-averaged. Bars across, waves or other means of symbol declaration for the time- and depth-averaging have been left away and will not be used in the following chapters either.
Learning goals in chapter 2:

Derivation and explanation of the 2D-depth-averaged shallow water equation and the turbulence.

- Describe the path of the initial equations to the depth-averaged shallow water equations.
- Which averaging do you know and how, or to which reason are they used?
- Gather all simplifications made.
- Which stresses occur at a fluid?
- What is a Newtonian fluid?
- What is the viscous stress tensor? What does it represent?
- What is an incompressible fluid? On what is the depth-averaged shallow water equation based?
- What is and what says the Stokes hypothesis?
- Why do you have to indicate the geographical latitude while doing a 2D simulation?
- What is the continuity equation?
- What is turbulence? (in own words)
- How are dynamic and kinematic viscosity connected together? Which unit do they have?
- What is the Reynolds-averaging? What simplifies it?
- Describe the energy cascades of the turbulence.
- What are coherent structures?
- What are apparent shear stresses? Sketch!
- Which 2 core approaches are there for the modelling of turbulence?
- What is assumed by the depth-averaging?
- How is the depth-averaging assigned to the equations?
- What means “vertical momentum exchange”?
- What is advection? And which is its equation? Find it in the depth-averaged shallow water equation?
- What is dispersion? And which is its equation? Find it in the depth-averaged shallow water equation?
- Where does the bed slope hide in the equation?
- Which parameters have to be calibrated at the 2D model? Why do you have to calibrate them and can not just insert them?
- Describe the following terms with your own words. Find out the units and simplify the terms where possible. Which of these terms are included in the depth-averaged shallow water equation (even if indirectly)?

\[
\frac{\partial z}{\partial t}, \frac{\partial}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial (A \cdot h)}{\partial x}, \frac{\partial w}{\partial t}, \frac{\partial u^2}{\partial x^2}
\]