2 FUNDAMENTALS OF MATHEMATICAL RIVER FLOW MODELLING

A solid understanding of the underlying mathematics is needed in order to be able to judge the adequacy of a mathematical method and its application limits. This knowledge helps to analyse how precisely the flow is described mathematically, which simplifying assumptions were made, and how they affect the result of the calculation. To help satisfying this need, the theoretical foundations of the two most common fundamental model approaches in engineering practise, the one-dimensional water level model and the two-dimensional shallow water model, will be discussed in detail in the following chapters.

2.1 ONE-DIMENSIONAL WATER LEVEL CALCULATIONS

The easiest mathematical method for calculating the flow situation in rivers is the water level calculation. It reduces the multi-dimensional real-world flow processes to a one-dimensional problem by assuming the flow cross-section and the flow velocity at a certain water level or, depending on the used approach, parts of the cross-section to be constant. The complex flow situation in the river is thus condensed to be a simple one-dimensional stream tube.

2.1.1 DEVELOPMENT OF A WATER LEVEL MODEL

To build a one dimensional flow model initially basic data are to be collected. Basically, preliminary considerations to the challenge and the goal of the modelling have always to be put in front of any data acquisition.

The structure of a classic 1D model runs in five steps:

1) Gathering all the input data.
2) Configuration of the 1D model.
3) Calibration of the 1D model.
4) Calculation/ Simulation for the design flood or other scenarios.
5) Evaluation and monitoring of the results.

At the gathering of all the input data it can be reverted to cards, literature, surveys and measurements.

- Number, location and extent of the profiles to be recorded
- Recording of the profile geometry
- Location and level data of inflows / outflows
- Grain size and structure of the river bed
- Structure and use of the floodplain
- Embankments, shore grouting and specialties
- Vegetation density, vegetation height, vegetation type and vegetation distance on the floodplain
- Gauge data of HQ, NQ and MQ – events with associated water levels
- Geometry, location and characteristics of bridges, culverts and weirs

While the configuration of the 1D model, all input data have to be transferred on the desired flow model and, where appropriate, assessments and simplifications have to be taken into account. These include the mapping of the cross sections, including the hydraulic parameters, the definition of the water course through the profiles, the definition of runoff events as well as the determination of all calculation settings such as simulation variants, flow-resistance act, parameters, etc.

As part of the calibration, which is usually carried out at first for the water bed under mean water and later for the floodplain under floodwater, the hydraulic parameters have to be calibrated. Here initially roughness, vegetation and hydraulic parameters are elected from past experience or references. After

an initial calculation the hydraulic parameters can be adjusted in a physically meaningful interval comparing the calculated results with the measurement. This calibration process will continue for as long as the results of calculating simulate the measurements as accurately as possible.
For the calculations of scenarios or design floods, the calibrated 1D model is the basis and (mostly) a reference state for studies. A steady control of the results by hand rough invoices or plausibility checks is recommended for each calculation. For this purpose especially the visualization of the results in a map and in the longitudinal section is suitable.

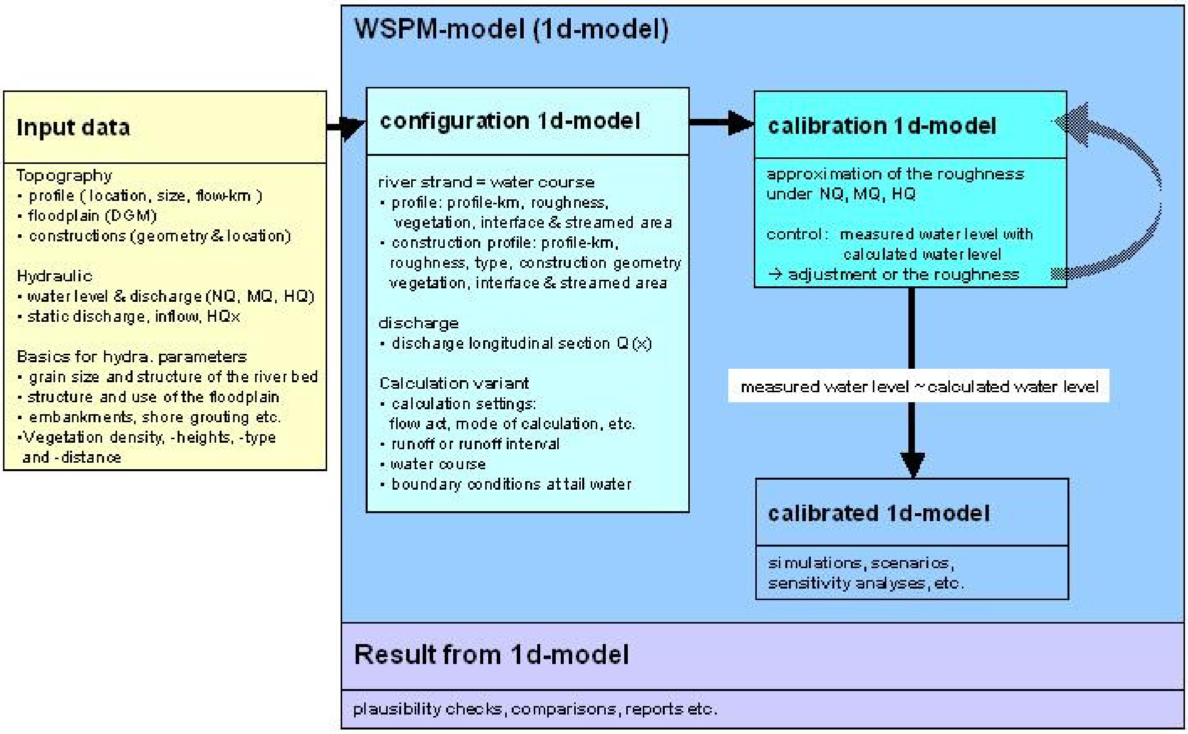

Figure 2-1: Development of an 1D-model for the calculation engine Kalypso 1d

2.1.1.1 SUPPLEMENTAL TERMINOLOGY

In the following the basic data of the one-dimensional water level calculation are defined and their significance is described.

Profile
A water profile is defined by the profile name (stations-km, profile state), about which the profile is referencable. A profile contains all the geometric and hydraulic data, which describe a cross section. Their content varies depending on the profile type (e.g. normal profile, bridge profile, weir profile). Regardless of the profile type, each profile contains the description of the land- or building geometry (in the y-z-level). This purely geometric description of the profile is supplemented by the hydraulic parameters such as bed roughness, vegetation, interface and floating zones, such as streamed part and the bankfull point.
The **growth parameters** ax, ay, and dp of the water profile after LINDER / PASCH will be additionally overlaid by the bed roughness. Hereby the growth parameters are defined as follows:

- ax = vegetation distance in the direction of flow,
- ay = vegetation distance crosswise to the direction of flow,

- dp = equivalent vegetation diameter (for trees equivalent to the trunk diameter, but for bushes much bigger).

For choosing the right parameters the BWK-bulletin 1, the DVWK-bulletin 220/1991 and other literature provide tables with growth parameters of different vegetation.
With the establishment of two interfaces per profile the so-called interface roughness between floodplain and river tube is considered after PASCH. The interfaces describe the momentum exchange between the floodplain and river tube and define the boundaries between the water flow in the water course and the significantly reduced water flow on the floodplain. Here the interface is also applied in the hydraulic calculation as a wall resistance between the river bed and floodplain. The interfaces have to be located at the transitions between extreme leap of roughness and/or increase of the ground level elevation. Basically, the two interfaces have to be arranged at the transition to the relevant roughness deprivation.
The streamed area indicates the area where the principle of linear momentum is applied. All areas outside the streamed area do not go in with a hydraulic calculation. The streamed area is to be defined in that manner that no distortion of the actual flow forces in the water course occurs. For example, troughs and branches adjacent to the stream course are not to be included in that area. Ideally, the streamed area should end at the outermost highest ground elevation point.

River strand
The consecutiveness of profiles along the water course is defined through the river strand. It also corresponds to the order of execution in the water level calculation. Moreover, the distances between the profiles in the river bed and on the floodplain are defined. The profile distances result from the user-entered values of the single stations. Also for each river strand a list of all the state's profiles is deposited. The order of the profile declaration is defined by the calculation direction opposite to the direction of the flow direction. The stationing direction is not fixed. However, it is basically recommended to fix the stations from the estuary in the direction of spring.

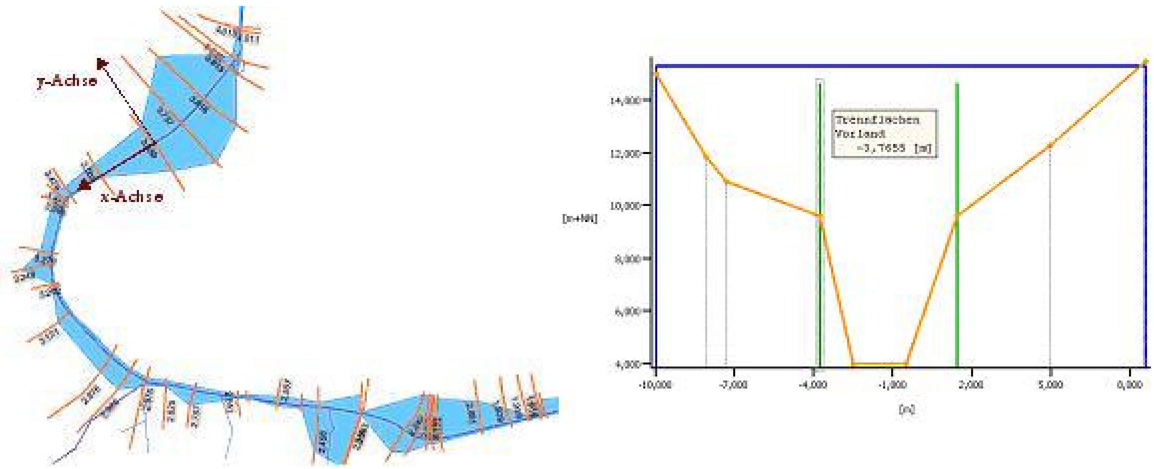

Figure 2-2: Left: river strand with water profiles in the plan view. Right: profile with interfaces (green) and streamed area (blue)

Discharge event
For a water level calculation the discharge has to be defined as a boundary condition. For this serves the discharge data set. Initially, a discharge data set is independent from the river strand and is only assigned to the river strand in the calculation variant. In this data set, the discharge can be established relative to a profile or even between two profiles. However, only those flow- kilometres have to get an discharge definition, where a discharge change occurs compared to the previous profile. Every discharge data set stand for exactly one discharge event. While the water level calculation the user can select the desired flow condition.

2.1.2 DERIVATION OF THE BASIC EQUATION

The basic mathematical equation of water level calculation, also called working equation, can be derived beginning with the energy theorem as well as with the momentum theorem. In the latter case, the continuity equation has to be considered. The derivation in this chapter is based on the momentum theorem. The following simplifying assumptions have to be made.

	Assumption		Details/ Explanations
1	quasi-steady flow:	$\frac{d(\vec{v})}{dt} = 0$	No modification over time
2	No momentum flow above water surface		No wind, no waves
3	Lateral momentum gain with constant velocity:	$\vec{v}_e = \text{konst.}$	Constant inflow velocity over the inflow cross section
4	Bernoulli-hypothesis:	$A_i \perp x - \text{Achse}$, $A_{i+1} \perp x - \text{Achse}$, $A_e \perp \vec{v}_e$	Flow vertical to the profile cross section
5	x-axis is river bed:	$\frac{d(z_{So})}{dx} = 0$	
6	Unidirectional flow:	$v_y \approx v_z \approx 0$	No secondary flow
7	Hydrostatic pressure distribution:	$p = p_o + \gamma \cdot (h - z)$	No turbulence (e.g. weirs)
8	Horizontal water level perpendicular to x-axis:	$\frac{d(z_{WSP})}{dy} = 0$	No super-elevation at the embankment
9	Continuum hypothesis	steady, connected	
10	Small change of flow along x-axis:	$\frac{\alpha_i}{\alpha_{i+1}} \approx 1$	No sudden widening/ constriction

Table 2-1: Assumptions for the derivation of the working equation.

The control volume of a river section and the coordinate system used in the momentum theorem is shown in Fig. 2-1. These initial conditions suffice for the definition of the momentum theorem.

The momentum rate of a material volume moved in space V^* is in the most general form:

$$\vec{F} = \frac{d\vec{I}}{dt} = \frac{d}{dt} \int_{V^*} \rho \, \vec{v} \, dV \qquad \text{Eq. 2-1}$$

with: $\vec{I} = \int_{V^*} \vec{v} \, dm = \int_{V^*} \rho \, \vec{v} \, dV$

V^* is an arbitrarily chosen material volume in the regarded moved continuum (Lagrange-model). The transition from a material volume to a rigid volume V (Euler-model) begins by exchanging the order of differentiation and integration. This is possible because the limits of integration are independent of the differentiation variable.

$$\frac{d}{dt} \int_{V^*} \rho \, \vec{v} \, dV = \int_{V} \frac{D}{Dt} (\rho \, \vec{v} \, dV) \qquad \text{Eq. 2-2}$$

The total change (total differential) of the Eularian field $\vec{v}$ is given by:

$$d\vec{v} = \frac{\partial \vec{v}}{\partial t}dt + \frac{\partial \vec{v}}{\partial x}dx + \frac{\partial \vec{v}}{\partial y}dy + \frac{\partial \vec{v}}{\partial z}dz \qquad \text{Eq. 2-3}$$

Using the Einstein summation convention, this expression can be written in tensor form as:

$$dv_i = \frac{\partial v_i}{\partial t}dt + \frac{\partial v_i}{\partial x_j}dx_j \qquad \text{Eq. 2-4}$$

If we take the material derivative (partial Lagrangeian derivative) of this equation, we obtain the vector form:

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \text{grad})\vec{v} \qquad \text{Eq. 2-5}$$

And in tensor form:

$$\frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j}v_j \qquad \text{Eq. 2-6}$$

The above equations consider that $\frac{Dt}{Dt} = 1$ and that $\vec{r}$ is the position vector with $\frac{D\vec{r}}{Dt} = \vec{v}$. If Eq. 2-6 is substituted into Eq. 2-2, the following equation for the rigid control volume is obtained:

$$F_i = \int_V \frac{D}{Dt}(\rho\, v_i\, dV) = \int_V \frac{\partial(\rho\, v_i)}{\partial t}dV + \int_V \rho\, v_j \frac{\partial v_i}{\partial x_j}dV \qquad \text{Eq. 2-7}$$

In order to get the second term of the right side of the equation into an area integral with help of the Gaussian integral theorem, the following transformation is done:

$$\int_V \frac{\partial v_i}{\partial x_j}v_j\, dV = \int_V \left(v_j \frac{\partial v_i}{\partial x_j} + v_i \underbrace{\frac{\partial v_j}{\partial x_j}}_{=0} \right) dV = \int_V \frac{\partial(v_i v_j)}{\partial x_j}dV \qquad \text{Eq. 2-8}$$

Adding the term $v_i \frac{\partial v_j}{\partial x_j}$ is allowed if we assume an incompressible fluid, since this term would be zero. The general form of the Gaussian integral theorem (divergence theorem) is stated as follows:

$$\int_V \text{div}\,\vec{a}\, dV = \int_A \vec{a} \cdot d\vec{A} \quad \text{or} \quad \int_V \frac{\partial a_i}{\partial x_i}dV = \int_A a_i\, dA_i \qquad \text{Eq. 2-9}$$

This integral theorem is not only valid for vectors, but can also be applied to tensors. The result is:

$$\int_V \operatorname{div} \overline{\overline{D}}\, dV = \int_A \overline{\overline{D}} \cdot d\vec{A} \quad \text{or} \quad \int_V \frac{\partial D_{ij}}{\partial x_i} dV = \int_A D_{ij}\, dA_i \qquad \text{Eq. 2-10}$$

The expression $v_i\, v_j$ or $\vec{v}\,\vec{v}$ respectively in Eq. 2-8 is a tensor, assuming the definition for tensor multiplication e.g.:

$$\vec{a}\,\vec{b} = \overline{\overline{D}} \quad \text{or} \quad a_i\, b_j = D_{ij}$$
$$\vec{a}\,\overline{\overline{D}} = \overline{\overline{\overline{E}}} \quad \text{or} \quad a_i\, D_{jk} = E_{ijk} \qquad \text{Eq. 2-11}$$

After this short survey of vector and tensor analysis, the 2. term on the right side of Eq. 2-7 can be transformed to a surface integral and we get:

$$\vec{F} = \int_V \frac{\partial(\rho\,\vec{v})}{\partial t} dV + \oint_A \rho\,\vec{v}\left(\vec{v}\cdot d\vec{A}\right) \quad or \quad F_i = \int_V \frac{\partial(\rho\, v_i)}{\partial t} + \int_A \rho\, v_i\, v_j\, dA_j \qquad \text{Eq. 2-12}$$

This equation is also called transport equation. It refers to the momentum as a property of a volume. It has been derived mathematically here, while some other teaching books choose a rather physical-graphical approach for derivation, e.g. *[Truckenbrodt, 1989]*.

$\vec{F}$ in Eq. 2-12 considers all the forces that act on the volumen V from the outside, the 1. term on the right side stands for the change in time of the momentum inside the control volume, and the 2. term on the right side stands for the momentum flow on the surface of the volume. The area vector $d\vec{A}$ can be characterized by ist size dA and its normal vector $\vec{n}$. The latter always points towards the outside in case of a closed area (surface). This leads to the following partition of the vector $d\vec{A}$:

$$d\vec{A} = \vec{n}\, dA = \vec{n}_x dA + \vec{n}_y dA + \vec{n}_z dA \qquad \text{Eq. 2-13}$$

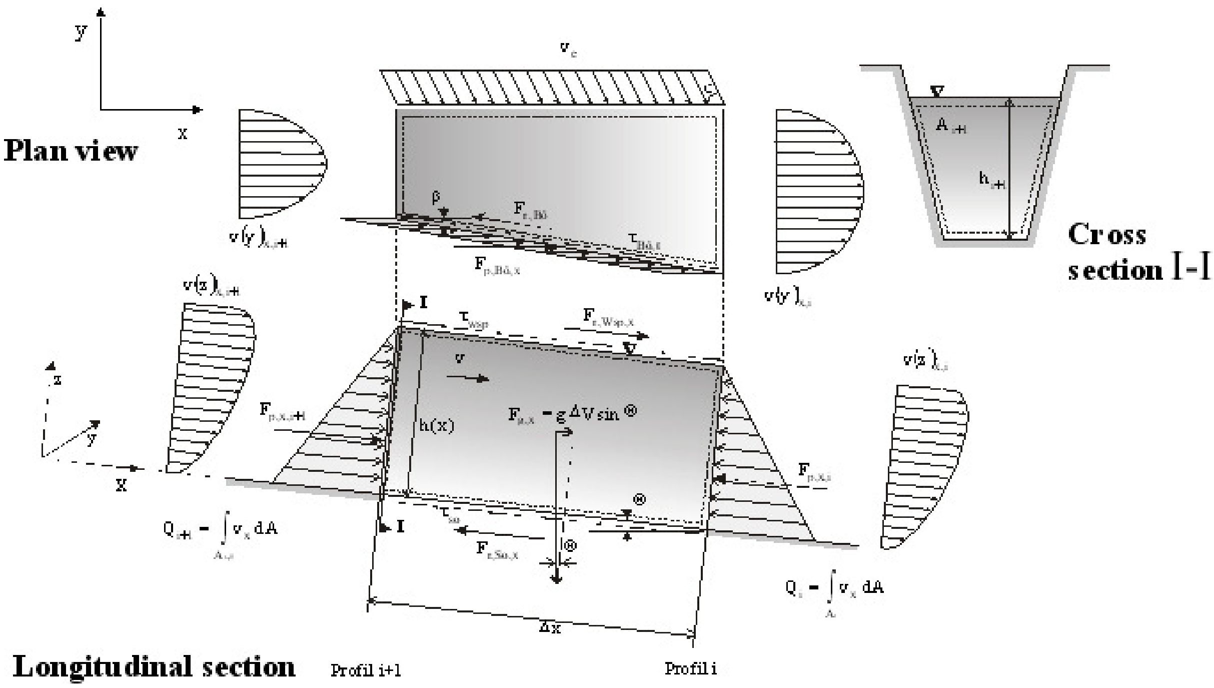

Figure 2-3: Definition sketch for the momentum theorem.

For simplification of the problem description, we assume a steady flow, so that the term depending on the time in Eq. 2-12 can be neglected (assumption 1 of Tab. 2-1). We further assume that the flow is preliminarily one-dimensional. The values of the transversal velocities v_y und v_z are thus almost zero. The general vectorial equation Eq. 2-12 can thus be reduced to the x-component.
The coordinate system is defined in the way that the x-axis is parallel to the river bed. The entry area A_{i+1} and the exit area A_i are perpendicular to the x-axis (assumptions 4, 5). The x-component of the momentum theorem is now:

$$F_x = \rho \int_{A_i} v_{x,i} \left(v_j \quad n_j \quad dA \right) + \rho \int_{A_e} v_{e,x} \left(\vec{v}_e \cdot d\vec{A}_e \right) + \rho \int_{A_{i+1}} v_{x,i+1} \left(v_j \quad n_j \quad dA \right) \qquad \text{Eq. 2-14}$$

applying the Einstein summation rule, which requires us to sum the terms with index j to j = 1,2,3. The normal vectors are $\vec{n}_i = (1,0,0)$ and $\vec{n}_{i+1} = (-1,0,0)$.

The term for the lateral inflow is:

$$\rho \int_{A_e} v_{x,e} \left(\vec{v}_e \cdot d\vec{A}_e \right) = + \int_{x_i}^{x_{i+1}} \rho \quad \cos\varphi \quad v_e \quad q_e \quad dx \qquad \text{Eq. 2-15}$$

With: φ = angle between x-axis and $\vec{v}_e$

For further simplification of the momentum equation, the momentum correction coefficient (Boussinesq coefficient) α' is introduced (definition):

$$\alpha' \equiv \frac{\int_A v_x^2 \, dA}{v^2 A} \qquad \text{Eq. 2-16}$$

With: v = mean flow velocity in cross section A.

With Eq. 2-15 and 2-16, the result of Eq. 2-14 is:

$$\begin{aligned} F_x &= \rho \;\; \alpha'_i \, v_i^2 \;\; A_i - \rho \;\; \alpha'_{i+1} \;\; v_{i+1}^2 \;\; A_{i+1} + \int_{x_i}^{x_{i+1}} \rho \;\; \cos\varphi \;\; v_e \;\; q_e \;\; dx \\ &= -\int_{x_i}^{x_{i+1}} \rho \;\; \frac{d\left(\alpha' \;\; v^2 \;\; A\right)}{dx} \, dx \; + \int_{x_i}^{x_{i+1}} \rho \;\; \cos\varphi \;\; v_e \;\; q_e \;\; dx \end{aligned} \qquad \text{Eq. 2-17}$$

It is to be noted that F_x is the sum of all the forces acting on the volume element, but only the components in direction of the x-axis. The individual force components that make up this sum will be explained in the following.

2.1.2.1 EXCURSE: THEORY ON THE IMPULSE CURRENT COEFFICIENT AND ENERGY FLUX COEFFICIENT

Riverine are often characterized by a classification of profiles and very different roughness ratios along the wetted perimeter. In this case, the flow is determined through two-dimensional flow. Investigations of *Pasche* show that an intense pulse exchange between floodplain and river tube occurs, which causes a significant loss of flow. Under these conditions, the premises of the work equation, such as horizontal water level and constant power level angles to the direction of flow is not met. Therefore, greatly due to the different flow velocities varies the dynamic energy level ($v^2 / 2g$) along a profile. Consequently, assuming a constant energy height at the profile, a water level cross fall occurs.
Investigations of Pasche show that the mean water level can be calculated precisely with the work equation, if the profile is separated into individual cross sections through fictitious interfaces, of which only the interfaces between the floodplain and river tube are with afflicted shear stress. Also the dynamic energy head (h_{dyn}) in the profile has to form the sum of the dynamic energy heads of all individual cross sections j, weighted with the flow share Q_j / Q is:
For the correction factor due to irregularity the velocity distribution over the cross section, there are two different approaches. Thus, the kinetic energy $v^2/2g$ is weighted with a kinetic energy coefficient $a \geq 1$, the so-called Coriolis-parameter. The kinetic energy can also be defined as a mass ρ, which per unit of time dt is being transported through a cross-sectional area dA.

$$\frac{d}{dt}\left(\rho \cdot \int_A v \, dA\right) \qquad \text{Eq. 2-18}$$

Thus, the kinetic energy of a fluid results in:

$$\frac{\alpha \cdot v^2}{2}\left(\rho \cdot \int_A v \, dA\right) = \frac{1}{2} \cdot \rho \cdot \alpha \cdot \int_A v^3 \, dA \qquad \text{Eq. 2-19}$$

Taking the mass conservation as a basis for the mean flow velocity v_m, the kinetic energy, which moves through a defined cross-section results to:

$$\frac{\alpha \cdot v_m^{\;2}}{2}\left(\rho \cdot v_m \int_A dA\right) = \frac{1}{2} \cdot \rho \cdot \alpha \cdot v_m^3 \int_A dA \qquad \text{Eq. 2-20}$$

From the combination of the two equations follows the following expression for the **Coriolis-parameter a**, if you divide a cross section in j sections:

$$\alpha = \frac{\int_A v^3 dA}{v_m^3 \int_A dA} \quad \text{or} \quad \alpha = \frac{\sum_j v_j^{\;3} \cdot A_j}{v_m^3 \cdot A} \qquad \text{Eq. 2-21}$$

Assuming the derivation is not due to the energy conservation but due to the conservation of momentum, you get the Boussinesq-parameter β:

$$\beta = \frac{\int_A v^2 dA}{v_m^2 \int_A dA} \quad \text{or} \quad \beta = \frac{\sum_j v_j^2 \cdot A_j}{v_m^2 \cdot A} \qquad \text{Eq. 2-22}$$

This approach results in the dynamic energy head h_{dyn}:

$$h_{dyn} = \frac{1}{2g}\sum v_j^3 A_j = \frac{Q^2}{2g}\alpha \qquad with \qquad \alpha = \frac{\sum v_j^3 A_j}{Q^3} = \frac{\sum v_j^3 A_j}{\left(\sum v_j A_j\right)^3} \qquad \text{Eq. 2-23}$$

For this relationship for α (= impulse current coefficient = Boussinesq- parameter) it becomes clear that this parameter represents a measure of the irregularity of the flow velocity in the profile. Subject to the underlying flow act, the velocity distribution factor α can be ascribed to a purely geometric parameters and coefficient of roughness. It can be chosen between the flow act of **Gauckler-Manning-Strickler** and that of **Darcy-Weisbach**.

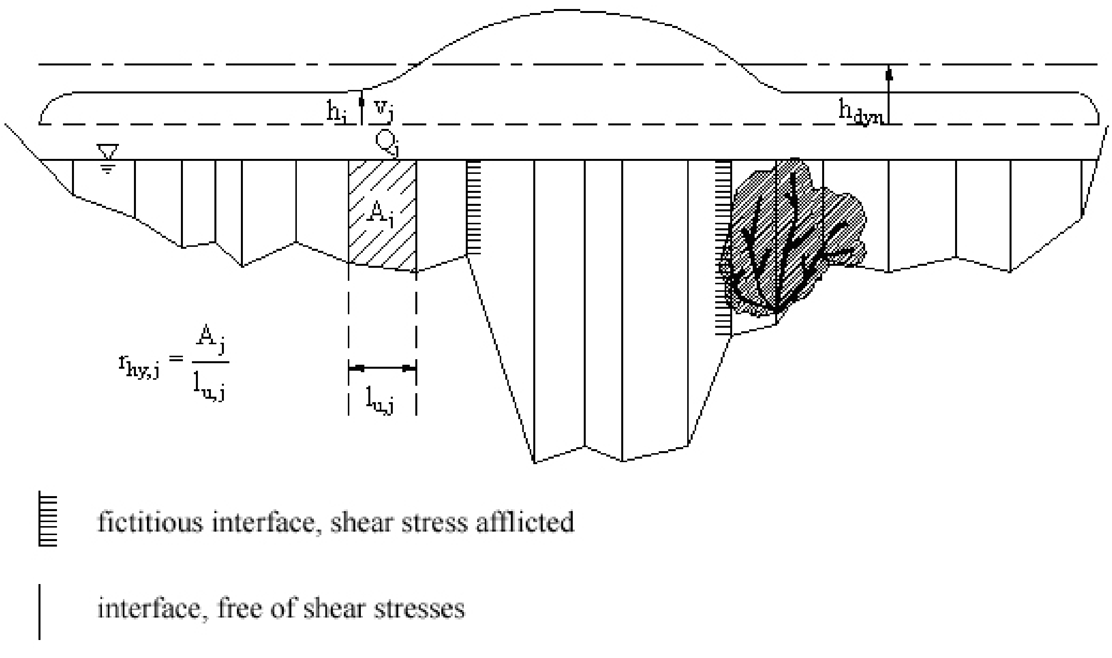

Figure 2-4: Subdivision of profiles for the calculation of h_{dyn}

About the flow law of Darcy-Weisbach follows:

$$v_j = \sqrt{\frac{8g}{\lambda_j} \cdot r_{hy,j} \cdot I_E} \qquad \text{Eq. 2-24}$$

The average speed in a cross-section is given by Jaegger to:

$$v_m = \frac{\sum_j v_j \cdot A_j}{A_{ges}} \qquad \text{Eq. 2-25}$$

For a subdivision of the cross section into three parts, the left floodplain (VL), the right floodplain (VR) and the river tube (FL), result the following description for the above-derived Boussinesq-parameter:

$$\beta = \frac{\left(v_{VR}{}^2 \cdot A_{VR}\right) + \left(v_{FL}{}^2 \cdot A_{FL}\right) + \left(v_{VL}{}^2 \cdot A_{VL}\right)}{\left[\left(v_{VR} \cdot A_{VR}\right) + \left(v_{FL} \cdot A_{FL}\right) + \left(v_{VL} \cdot A_{VL}\right)\right]^2 / A} \qquad \text{Eq. 2-26}$$

The flow velocity in the various sections will be replaced by the flow act of Darcy-Weisbach. The flow velocity can be reduced. Under the assumption that I_E is constant for all sub-sections it follows:

$$\beta = \frac{A_{ges} \cdot \left(\left(\left(\frac{r_{hy,VR}}{\lambda_{VR}}\right) \cdot A_{VR}\right) + \left(\left(\frac{r_{hy,FL}}{\lambda_{FL}}\right) \cdot A_{FL}\right) + \left(\left(\frac{r_{hy,VL}}{\lambda_{VL}}\right) \cdot A_{VL}\right)\right)}{\left[\left(A_{VR}\sqrt{\frac{r_{hy,VR}}{\lambda_{VR}}}\right) + \left(A_{FL}\sqrt{\frac{r_{hy,FL}}{\lambda_{FL}}}\right) + \left(A_{VL}\sqrt{\frac{r_{hy,VL}}{\lambda_{VL}}}\right)\right]^2} \qquad \text{Eq. 2-27}$$

With analog adjustment means that the Coriolis parameter:

$$\alpha = \frac{\sum_j v_j{}^3 \cdot A_j}{v_m^3 \cdot A} = \frac{A_{ges}{}^2 \cdot \left(\left(\left(\frac{r_{hy,VR}}{\lambda_{VR}}\right)^{3/2} \cdot A_{VR}\right) + \left(\left(\frac{r_{hy,FL}}{\lambda_{FL}}\right)^{3/2} \cdot A_{FL}\right) + \left(\left(\frac{r_{hy,VL}}{\lambda_{VL}}\right)^{3/2} \cdot A_{VL}\right)\right)}{\left[\left(A_{VR}\sqrt{\frac{r_{hy,VR}}{\lambda_{VR}}}\right) + \left(A_{FL}\sqrt{\frac{r_{hy,FL}}{\lambda_{FL}}}\right) + \left(A_{VL}\sqrt{\frac{r_{hy,VL}}{\lambda_{VL}}}\right)\right]^3} \qquad \text{Eq. 2-28}$$

2.1.2.2 EXCURSE: THEORY OF THE FRICTION LOSSES

The friction loss h_r is a continuing loss, which can be gained through the integration of the section Δx.

$$h_r = \int_{x_i}^{x_i+1} I_r(x)dx \qquad \text{Eq. 2-29}$$

After NAUDASCHER (1992) this integral can only be calculated with a gradual change of geometric and sufficiently small step size Δx using the trapezoid formula in good approximation. (Figure 2-5)

$$h_r = \Delta x \cdot \frac{1}{2} \cdot \left(I_{r,i} + I_{r,i+1}\right) \qquad \text{Eq. 2-30}$$

For sudden geometric changes or strong gradient changes, the approach of the trapezoidal formula leads to an over-estimation of the friction loss, if the profile distance is far too large and the calculation profile is situated directly at the height of the discontinuity or at the breakpoint of the hydraulic gradient. In the water level calculation, this either means that the basic equation is not a solution for the profile upstream or delivers a physically absurd rise in the water level upstream. In this case, the profile has to be reduced and the calculation profile has to be situated in the area of the discontinuity or the hydraulic gradient breakpoint, so that the arithmetic mean of the gradient of friction at the profile i and i +1is equal to the actual friction loss.

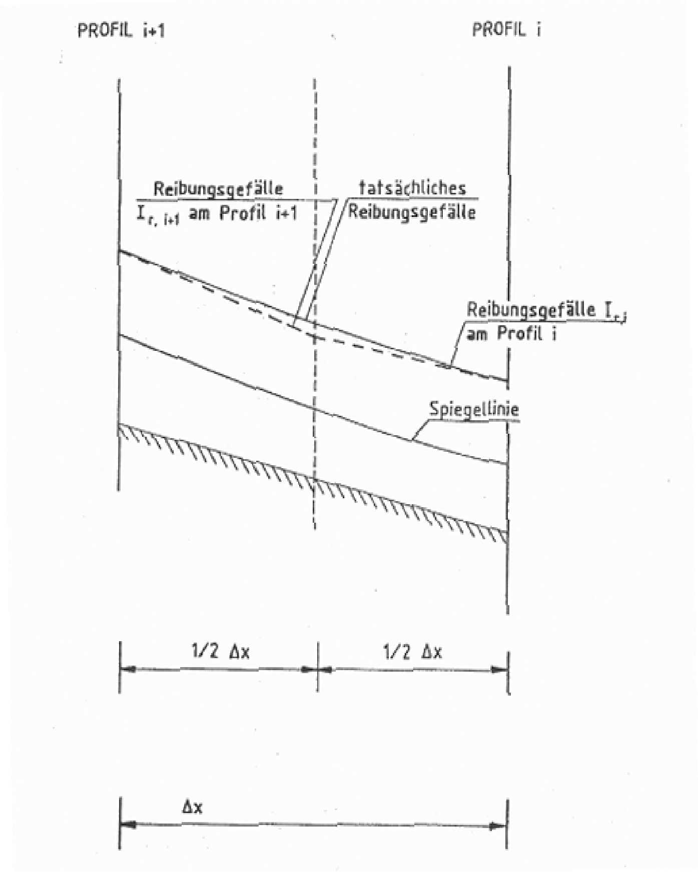

Figure 2-5: Model assumptions for friction gradient

The flows in rivers are almost always turbulent. The gradient of friction I_r behaves in this case proportional to the square of the flow velocity and is recorded on the following relationship:

$$I_r = \frac{\lambda}{d_{hy}} \cdot \frac{Q^2}{A^2} \cdot \frac{1}{2g}$$ Eq. 2-31

The parameter λ is dimensionless. Nonetheless, he is not a real constant, but dependent on the surface structure of the wetted cross-section. This can vary considerably along the wetted perimeter $I_{U,ges}$, so that in this case λ adopts different values in the profile. After EINSTEIN (1934) these may be overlaid to a total linear resistance, if v_j/v ~ 1,0 and therefore exists a compact channel.

$$\lambda_{ges} = \frac{\sum_{j=1}^{n} \lambda_j \cdot l_{u,j}}{l_{u,ges}}$$ Eq. 2-32

with: j=1, 2, ..., n

n= number of roughness sections j in the cross-section with v_j~v

$l_{u,j} = \frac{A_j}{r_{hy,j}}$ = wetted perimeter for roughness sections j

$l_{u,ges} = \frac{A_{ges}}{r_{hy,ges}}$ = wetted perimeter for the total cross-section

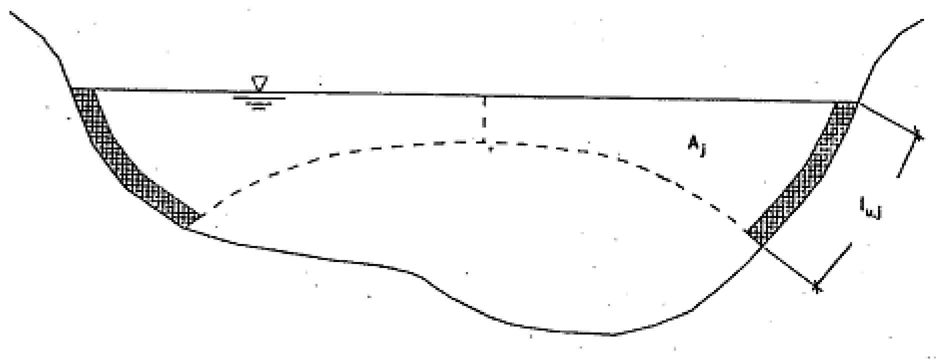

Figure 2-6: Water profile with sectional varying roughness

Thereby, for each roughness sections one subarea A_j is assigned whose interfaces are free of shear stress (the interfaces are orthogonal to the isotachs). The size or the hydraulic radius of this subarea is proportional to the friction coefficient of the single roughnesses:

$$\frac{\lambda_j}{\lambda_{ges}} = \frac{r_{hy,j}}{r_{hy,ges}}$$ Eq. 2-33

With this overlay approach and the assumption of a flow direction perpendicular to the horizontal water level the friction gradient I_r can be calculated with sectional variable bed-roughnesses to:

$$I_r = \frac{1}{\left(\sum A_j \left(r_{hy,j}/\lambda_j\right)^{o,5}\right)^2} \cdot \frac{Q^2}{8g} \qquad \text{Eq. 2-34}$$

In river/floodplain flows and water courses with streamed woody zones the bed roughness and water depth vary so much along the wetted perimeter that large differences in the flow velocities between the cross-sections occur. In this case, the interfaces between these cross-sections are shear stress afflicted and have to be included in the scope of the wetted perimeter. (Figure 2-7)

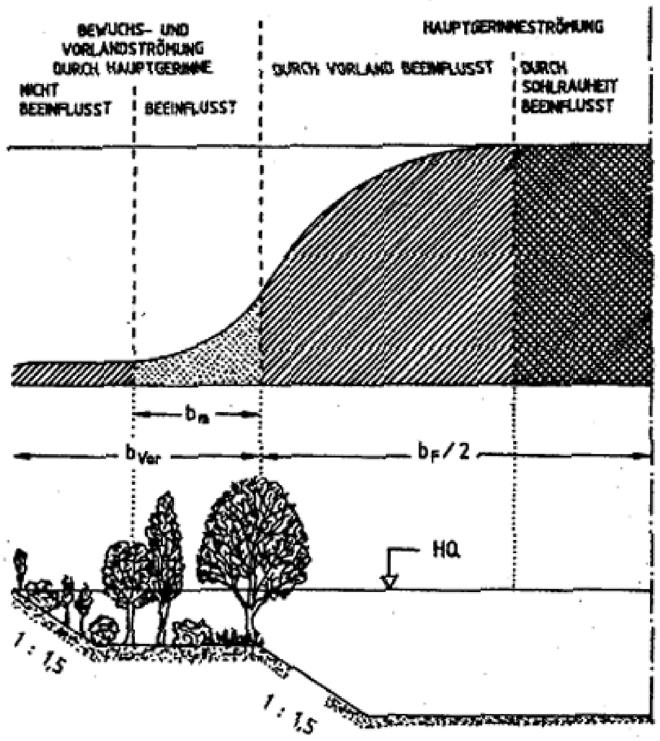

Figure 2-7: Subdivision of the cross-sections and flow of the river and floodplain

c

$$\lambda_F = \frac{\lambda_{T,li} \cdot h_{T,li} + \lambda_{T,re} \cdot h_{T,re} + \sum_{j=1}^{n} \lambda_{So,j} \cdot l_{u,j}}{\sum_{j=1}^{n} I_{u,j} + h_{T.li} + h_{T,re}} \qquad \text{Eq. 2-35}$$

with: λ_T = frictional resistance in the right (re) or left (li) interface
h_T = water depth in the right (re) or left (li) interface
j = 1, 2, ..., n = number of cross-sections with different bed roughnesses in the river tube

For the floodplain and vegetation area, this fictitious interface will not be included in the wetted perimeter. However, in the event of a streamed woody vegetation we have the specific situation that in one section two frictional resistance occur, which, according to LINDNER (1982) interfere with each other to (Figure 2-8):

$$\lambda_j = \lambda_{So,j} + \lambda_{p,j} \qquad \text{Eq. 2-36}$$

with: $\lambda_{So,j}$ = flow resistance coefficient at the river bed in the sub-cross-section j
$\lambda_{p,i}$ = flow resistance coefficient due to streamed vegetation in the sub-cross-section j

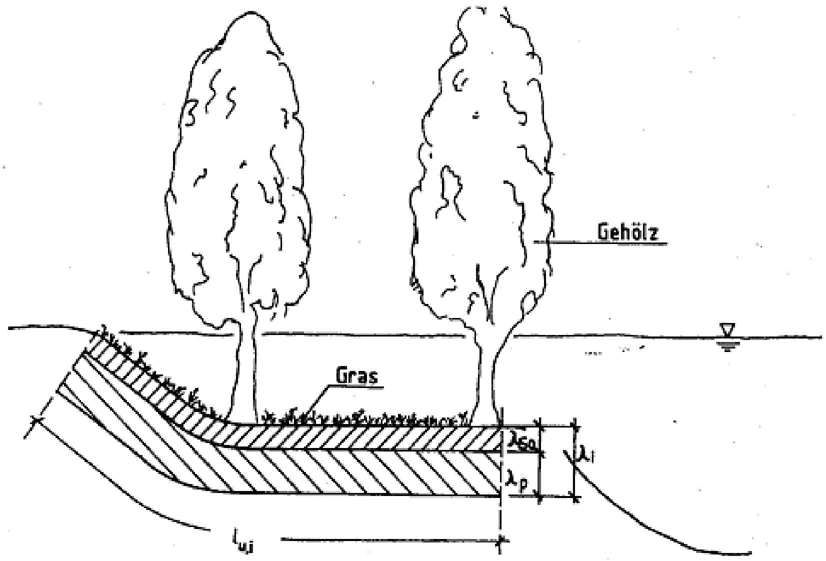

Figure 2-8: ratios of roughness in with trees vegetated floodplains

There wise , the mean flow resistance λ_{vor} on the floodplain or in the vegetation zone can be calculated to:

$$\lambda_{Vor} = \frac{\sum_{j=1}^{n} \left(\lambda_{So,j} + \lambda_{p,j} \right) \cdot l_{u,j}}{\sum_{j=1}^{n} l_{u,j}} \qquad \text{Eq. 2-37}$$

with: j = 1, 2, ..., n = number of sections with different roughness or water depth on the floodplain or vegetation zone

On this basis the frictional gradient I_r can be calculated for a roughness- and cross-section divided water profile to:

$$I_r = \frac{1}{\left[A_{Vor,li} \left(r_{hy,Vor,li} / \lambda_{Vor,li} \right)^{0,5} + A_{Vor,re} \left(r_{hy,Vor,re} / \lambda_{Vor,re} \right)^{0,5} + A_F \left(r_{hy,F} / \lambda_F \right)^{0,5} \right]^2} \cdot \frac{Q^2}{8g} \quad \text{Eq. 2-38}$$

with: $r_{hy,Vor} = \frac{A_{Vor}}{l_{u,Vor}}$ = hydraulic radius at the floodplain [m]

$r_{hy,F} = \frac{A_F}{l_{u,So,F} + h_{T,li} + h_{T,re}}$ = hydraulic radius in the river [m]

$l_{u,So,F}$= length of the wetted perimeter along the river bed and vegetation zone of the wood-free river tube

$l_{u,Vor}$= length of the wetted perimeter along the bed of the floodplain or the vegetation zone, respectively for the right (re) or left (li) of the floodplain or vegetation zone

2.1.2.3 DESCRIPTION OF ALL FORCE COMPONENTS F_X ACTING ON THE VOLUME:

The forces that act on the water volume V are composed of the surface forces $\vec{F}_O$ acting on the outside and the mass forces or volume forces $\vec{F}_V$. While the mass forces can be seen as forces that are implied by outside factors, the surface forces depend on the movement state of the liquid.

The surface forces in channel flow that may occur are:

- Frictional force: $\vec{F}_r$
- Pressure force: $\vec{F}_p$
- Form drag: $\vec{F}_w$

The mass forces are:

- Gravitational force: $\vec{F}_g$
- Coriolis force: $\vec{F}_c$

The Coriolis forces and other forces like the centrifugal force and forces due to changes in the thermal state that have not been mentioned above are generally neglected in one-dimensional water level calculations. Neglected as well is the friction at the water surface (e.g. by wind influence). However, this is introduced a little later, so that we will keep the term for the friction at the surface for a while. The equations for the individual forces will be derived in the following:

2.1.2.3.1 SURFACE FORCES:

Matter resists a movement of its molecules against each other. Also water has this property, commonly called viscosity or tenacity. It is shown in the form of normal and shear stress forces. While the normal stress acts perpendicular to the water surface, just like the liquid pressure, the shear stress is a tangential force. It is formed e.g. by adhesion at walls and the viscosity of the water. The value of the shear stress is a measure for the frictional force of the liquid. In a plane shear flow it is proportional to the velocity gradient du/dy.

$$\tau = \mu \frac{du}{dy} \qquad \text{Eq. 2-39}$$

This elementary law of liquid friction is also called Newtons friction law. The factor of proportionality μ is a material constant of the liquid which depends much on temperature, and which is called viscosity of a liquid.

Eq. 2-39 corresponds in form and content to the elasticity law of an elastic, firm body. Hooke's law states that the shear stress of an elastic body is proportional to the value of deformation.

$$\tau = G \frac{ds}{dy} \qquad \text{Eq. 2-40}$$

The shear modulus G in this law corresponds to the dynamic viscosity μ and the deformation ds/dy in direction of the x-axis corresponds to the deformation velocity du/dy.

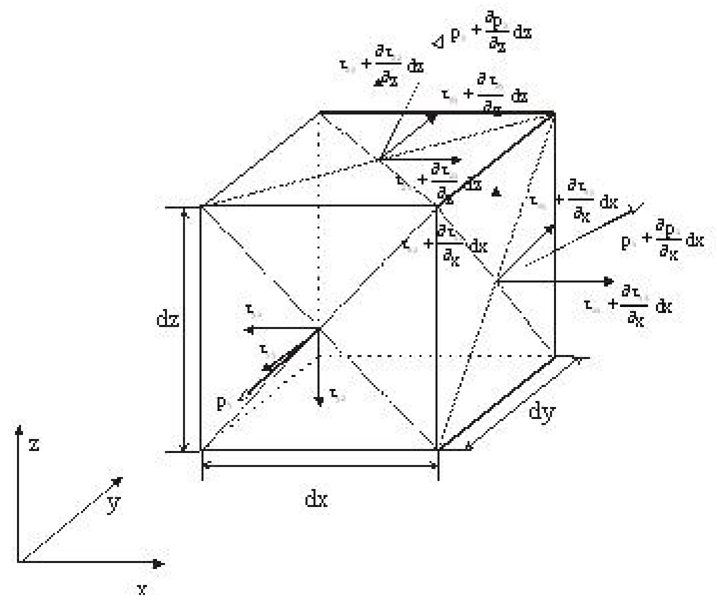

Figure 2-9: Volume element with its stresses.

For better modelling and evaluation of the surface forces that act on the water volume dV, we imagine a cuboid body with the edges dx, dy, dz (see Figure 2-9). The resulting forces act on this body:

$$\vec{p}_x\ ,\ \vec{p}_x + \frac{\partial \vec{p}_x}{\partial x}dx;\quad \vec{p}_y\ ,\ \vec{p}_y + \frac{\partial \vec{p}_y}{\partial y}dy;\quad \vec{p}_z\ ,\ \vec{p}_z + \frac{\partial \vec{p}_z}{\partial z}dz \qquad \text{Eq. 2-41}$$

If these forces are evaluated over the volume dV, the total surface forces result in:

$$d\vec{F}_O = \left(\frac{\partial \vec{p}_x}{\partial x} + \frac{\partial \vec{p}_y}{\partial y} + \frac{\partial \vec{p}_z}{\partial z}\right) dV \qquad \text{Eq. 2-42}$$

These stress vectors are closely connected to the direction of the area element they act on. This fact is obvious in resting, friction-less fluids, that are not subject to shear stress at all. Only normal stresses, that are perpendicular to the area element, can occur in these systems. In this case, assuming a hydrostatic stress state, the normal stresses are only composed of the pressure stresses $\vec{\sigma}_p$ that are directed towards the inside (pressure forces, not traction forces) of the area element dA, with its normal vector $\vec{n}$ pointing to the outside by definition.

$$d\vec{F}_O = \vec{\sigma}_p dA = -p\,\vec{n}\,dA \qquad \text{Eq. 2-43}$$

Since the value σ_p of a stress vector in a point does not depend on the orientation of the area element, it can be described by a single, scalar field, the local pressure p, as:

$$\sigma_p = \tau_{xx} = \tau_{yy} = \tau_{zz} = -p \qquad \text{Eq. 2-44}$$

The stress vector $\vec{\sigma}_p$ however depends on the orientation of the area element, which is expressed in Eq. 2-43 by the normal vector $\vec{n}$. This dependence is also valid for friction-affected flows. In contrast to friction-less fluids, the stress vectors also depend on the orientation of the area element they act on, so that the relation of the vectors $\vec{p}$ and $\vec{n}$ can not be described by a scalar field, but only by a stress tensor:

$$\vec{p}\,dA = (\overline{\overline{T}} \cdot \vec{n})\,dA \qquad \text{Eq. 2-45}$$

The stress tensor $\overline{\overline{T}}$ is obtained with reference to Eq. 2-42. The appearing vectors $\vec{p}_x, \vec{p}_y$ and $\vec{p}_z$ can be decomposed into their single components:

$$\begin{aligned}
\vec{p}_x &= p_x(e_x)\,\vec{e}_x + p_x(e_y)\,\vec{e}_y + p_x(e_z)\,\vec{e}_z \\
\vec{p}_y &= p_y(e_x)\,\vec{e}_x + p_y(e_y)\,\vec{e}_y + p_y(e_z)\,\vec{e}_z \\
\vec{p}_z &= p_z(e_x)\,\vec{e}_x + p_z(e_y)\,\vec{e}_y + p_z(e_z)\,\vec{e}_z
\end{aligned} \qquad \text{Eq. 2-46}$$

The braces that are used for the components of the stress vector are only supposed to show its dependence of the orientation vector. The decomposition can also be written as follows, referring to Figure 2-9:

$$\begin{aligned}
\vec{p}_x &= \tau_{xx}\,\vec{e}_x + \tau_{xy}\,\vec{e}_y + \tau_{xz}\,\vec{e}_z \\
\vec{p}_y &= \tau_{yx}\,\vec{e}_x + \tau_{yy}\,\vec{e}_y + \tau_{yz}\,\vec{e}_z \\
\vec{p}_z &= \tau_{zx}\,\vec{e}_x + \tau_{zy}\,\vec{e}_y + \tau_{zz}\,\vec{e}_z
\end{aligned} \qquad \text{Eq. 2-47}$$

The first element of the double index refers to the axis the area element is perpendicular to, and the second element to the axis, the stress vector points to. By comparing 2-47 and Eq. 2-45, we obtain a formula for the stress tensor $\overline{\overline{T}}$:

$$\overline{\overline{T}} = \begin{Bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{Bmatrix} \qquad \text{Eq. 2-48}$$

The stress tensor is symmetric, since the material properties of incompressible fluids are not depending on the orientation (isotropy). This means that the tangential stresses with the same indices in a different order are the same:

$$\tau_{xy} = \tau_{yx}\,; \quad \tau_{xz} = \tau_{zx}\,; \quad \tau_{yz} = \tau_{zy} \qquad \text{Eq. 2-49}$$

The stress tensor is also valid for friction-less fluids. In this case all the tangential stresses are equal to zero. The only remaining stresses are normal stresses, which all have the same value and correspond to the hydrostatic pressure:

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = -p \qquad \text{Eq. 2-50}$$

If the stress tensor from Eq. 2-48 is substituted intoEq. 2-42, the formula for the surface force is:

$$d\vec{F}_O = \begin{Bmatrix} \frac{\partial \tau_{xx}}{\partial x} & \frac{\partial \tau_{xy}}{\partial y} & \frac{\partial \tau_{xz}}{\partial z} \\ \frac{\partial \tau_{yx}}{\partial x} & \frac{\partial \tau_{yy}}{\partial y} & \frac{\partial \tau_{yz}}{\partial z} \\ \frac{\partial \tau_{zx}}{\partial x} & \frac{\partial \tau_{zy}}{\partial y} & \frac{\partial \tau_{zz}}{\partial z} \end{Bmatrix} \begin{Bmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{Bmatrix} dV \qquad \text{Eq. 2-51}$$

Or in vector form, integrating over the total volume V:

$$\vec{F}_O = \int_V d\vec{F}_o dV = \int_V \operatorname{div} \overline{\overline{T}}\, dV \qquad \text{Eq. 2-52}$$

The volume integral can be transformed to a surface integral by applying the Gaussian integral theorem. This yields:

$$\vec{F}_O = \int_V \operatorname{div} \overline{\overline{T}}\, dV = \oint_A \overline{\overline{T}} \cdot d\vec{A} \qquad \text{Eq. 2-53}$$

We now have to transfer the definition of the surface stresses of the volume element (see Figure 2-9) to the control volume of the river section between sections i and i+1, which is shown in Figure 2-3. This could be seen as moving from an infinitesimal point of view to a macroscopic point of view. At this time we also presume the assumptions in Tab. 2-1 that refer to the properties of the flow being parallel to the river bed and its direction being orthogonal. The following expression for the sum of the surface forces in direction of the x-axis of the control volume is obtained with reference to Eq. 2-53:

$$\begin{aligned} F_{O,x} = & \int_{A_i} \tau_{xx,i}\, dA - \int_{A_{i+1}} \tau_{xx,i+1}\, dA - \int_{A_{So}} \tau_{zx}\, dA + \int_{A_{Wsp}} \tau_{zx}\, dA \\ & - \int_{A_{Bö,r}} \tau_{yx}\, dA - \int_{A_{Bö,r}} \tau_{yy,x}\, dA + \int_{A_{Bö,l}} \tau_{yx}\, dA + \int_{A_{Bö,l}} \tau_{yy,x}\, dA \end{aligned} \qquad \text{Eq. 2-54}$$

with: A_{So} = area of the wet river bed in volume V.
A_{Wsp} = area of the water surface in volume V.
$A_{Bö,l}$ = area of the left slope in volume V.
$A_{Bö,r}$ = area of the right slope in volume V.

In order to be able to specify Eq. 2-54 further, we divide the components of the equation into shear stresses (friction at the bed, at the water surface and at the slopes) and normal stresses (pressure stresses at the sections i and i+1 and at the slopes). Firstly we look at the shear stresses:

If the cross section changes in direction of flow

$$\frac{db(z)}{dx} \neq 0 \qquad \text{Eq. 2-55}$$

The slope plane is not in direction of the x-axis, so that the shear stress $\tau_{Bö}$ does not point in direction of the shear stresses in the main planes and thus has to be corrected by the cosine of the enlargement angle β.

$$\int_{A_{Bö}} \tau_{yx}\, dA_y = \int_{A_{Bö}} \tau_{Bö,t} \cos\beta \, dA_y = \int_{A_{Bö,y}} \tau_{Bö,t} \cos^2\beta \, dA \qquad \text{Eq. 2-56}$$

with $\beta =$ enlargement angle of the slope measured towards the x-axis.
$A_{Bö,y} =$ slope area projected to the x-axis. In analogy to the indices of the stresses, it has the index y since the area is perpendicular to the y-axis.

The following descriptors stand for the shear stresses at the bed and at the surface:

$$\int_{A_{So}} \tau_{zx}\, dA = \int_{A_{So}} \tau_{So}\, dA \quad \wedge \quad \int_{A_{Wsp}} \tau_{zx}\, dA = \int_{A_{Wsp}} \tau_{Wsp}\, dA \qquad \text{Eq. 2-57}$$

For further simplification we combine the shear stresses at the walls, i.e. at the slopes and the bed, and introduce the friction slope I_R. The shear stresses at the walls can now be described as:

$$\begin{aligned} -\int_{A_{Wand}} \tau_{Wand}\, dA &= -\int_{A_{So}} \tau_{So}\, dA - \int_{A_{Bö,r,y}} \tau_{Bö,t} \cos^2\beta_r \, dA - \int_{A_{Bö,l,y}} \tau_{Bö,t} \cos^2\beta_l \, dA \\ &\equiv +\int_{x_i}^{x_{i+1}} \rho \; g \; I_R \; A(x)\, dx \end{aligned} \qquad \text{Eq. 2-58}$$

Secondly, we look at the surface stress with the same double index, i.e. the normal stresses. We introduce a separation of the pressure stresses, which means that the normal stress is composed of the sum of the negative pressure and a deviator stress:

$$\tau_{xx} = \sigma_x - p \quad \wedge \quad \tau_{yy} = \sigma_y - p \quad \wedge \quad \tau_{zz} = \sigma_z - p \qquad \text{Eq. 2-59}$$

The value of the deviator stresses $\sigma_x, \sigma_y, \sigma_z$ due to the deviation of the hydrostatic pressure distribution are very small for small secondary flows (assumption 10 in Tab. 2-1), so they can be neglected in good approximation:

$$\sigma_x \approx 0 \quad \wedge \quad \sigma_y \approx 0 \quad \wedge \quad \sigma_z \approx 0 \qquad \text{Eq. 2-60}$$

The only normal stress that remains is the pressure stress p. Assuming a hydrostatic pressure distribution (assumption 7), it is a function of the water depth:

$$p = p_o + \rho\, g\,(h - z)\cos\Theta \qquad \text{Eq. 2-61}$$

with $p_0 =$ atmospheric pressure.
$\cos\Theta =$ correctional factor due to bed slope.

The atmospheric pressure p_o on the surface is the same everywhere, but the formula for the momentum only adapts to changes of the volume V, the factor p_o in Eq. 2-61 can be dropped. With regard to assumptions 4,5,6 and 7, the x-component of the normal stresses at the volume element is:

$$\begin{aligned}
&\int_{A_i} \tau_{xx}\; dA - \int_{A_{i+1}} \tau_{xx}\; dA - \int_{A_{Bö,\,r}} \tau_{yy,\,x}\; dA + \int_{A_{Bö,\,l}} \tau_{yy,\,x}\; dA = \\
&\quad - \int_{A_i} \rho\; g\; (h_i - z)\; \cos\Theta\; dA \\
&\quad + \int_{A_{i+1}} \rho\; g\; (h_{i+1} - z)\; \cos\Theta\; dA \\
&\quad + \int_{A_{Bö,\,r+l,\,x}} \rho\; g\; (h - z)\; \cos\Theta\; dA
\end{aligned} \qquad \text{Eq. 2-62}$$

The last term of this equation shows the pressure forces along the channel slopes that occur when the cross section changes from A_i to A_{i+1}. Assuming a horizontal water surface in direction of the y-axis, this term becomes:

$$\int_{A_{Bö,\,r+l,\,x}} \rho\; g\; (h - z)\cos\Theta\; dA = -\int_{x_i}^{x_{i+1}} \int_0^h \rho\; g(h - z)\; \cos\Theta\; \left(\frac{db(z)}{dx}\right)\; dz\; dx \qquad \text{Eq. 2-63}$$

If we summarize the results for the shear and normal stresses from Eq. 2-58, Eq. 2-62 and Eq. 2-53, the equation Eq. 2-54 can be written as:

$$\begin{aligned}
F_{O,\,x} = &-\int_{A_i} \rho\; g\; (h - z)\; \cos\Theta\; dA + \int_{A_{i+1}} \rho\; g\; (h - z)\; \cos\Theta\; dA \\
&- \int_{x_i}^{x_{i+1}} \int_0^h \rho\; g(h - z)\; \cos\Theta\; \left(\frac{db(z)}{dx}\right)\; dz\; dx \\
&+ \int_{A_{Wsp}} \tau_{Wsp}\; dA + \int_{x_i}^{x_{i+1}} \rho\; g\; I_R\; A(x)\; dx
\end{aligned} \qquad \text{Eq. 2-64}$$

2.1.2.3.2 MASS FORCES:

The mass force $\vec{F}_g$ can be expressed as a gradient of the gravity potential Γ:

$$\vec{F}_g = \rho \int_V \mathrm{grad}(\Gamma)\, dV \qquad \text{Eq. 2-65}$$

The gravitation potential corresponds to the vector field $\vec{g}$ of the gravitation constant. Since the river bed has a gradient angle Θ, the vector field has to be stated according to the rotated coordinate system, so that there is an inertial force due to gravitation not only in direction of the z-axis, but also in direction of the x-axis:

$$\vec{g} = \mathrm{grad}(\Gamma) = \frac{\partial \Gamma}{\partial x}\vec{e}_x + \frac{\partial \Gamma}{\partial z}\vec{e}_y + \frac{\partial \Gamma}{\partial z}\vec{e}_z = \frac{\partial(g \sin\Theta\, x)}{\partial x}\vec{e}_x + \frac{\partial(0 \cdot y)}{\partial y}\vec{e}_y + \frac{\partial(g \cos\Theta\, z)}{\partial z}\vec{e}_z \qquad \text{Eq. 2-66}$$

The x-component of the inertial force is thus:

$$F_{g,x} = \rho \int_V \frac{\partial(g \ \sin\Theta \ x)}{\partial x} dV = -\int_{x_i}^{x_{i+1}} \rho \ g \ \sin\Theta \ A(x) \ dx$$
$$= -\int_{x_i}^{x_{i+1}} \rho \ g \ \tan\Theta \cos\Theta \ A(x) \ dx \qquad \text{Eq. 2-67}$$

2.1.2.4 COMPILATION OF ALL FORCE COMPONENTS ACTING ON THE VOLUME

So far, the surface forces (Eq. 2-64) and the inertial forces (Eq. 2-67) have been stated for the control volume of the river section. The result according to Eq. 2-17 is the total momentum balance at the control volume V (in direction of the x-axis):

$$\begin{aligned}\sum F_x &= -\int_{x_i}^{x_{i+1}} \rho \frac{d\left(\alpha' \; v^2 \; A\right)}{dx} dx + \int_{x_i}^{x_{i+1}} \rho \; \cos\varphi \; v_e \; q_e \; dx \\ &= -\int_{A_i} \rho g \; (h-z) \; \cos\Theta \; dA + \int_{A_{i+1}} \rho g \; (h-z) \; \cos\Theta \; dA \\ &\quad - \int_{x_i}^{x_{i+1}} \int_0^h \rho g (h-z) \; \cos\Theta \; \left(\frac{db(z)}{dx}\right) dz \; dx \\ &\quad + \int_{A_{Wsp}} \tau_{Wsp} \; dA + \int_{x_i}^{x_{i+1}} \rho \; g \; I_R \; A(x) \; dx \\ &\quad - \int_{x_i}^{x_{i+1}} \rho \; g \; \tan\Theta \cos\Theta \; A(x) \; dx\end{aligned} \qquad \text{Eq. 2-68}$$

For further simplification of the momentum formula, the following assumptions are made:

- 9. assumption: continuous change of the stream variables along the x-axis.
- Neglecting of shear stresses at the water surface.

The implication is a linear pressure, bed shear stress and velocity change along the x-axis, so that we can state a discrete formula like:

$$v_i = v_{i+1} + \frac{dv}{dx} \Delta x \,, \qquad \text{Eq. 2-69}$$

$$A_i = A_{i+1} + \frac{dA}{dx} \Delta x \qquad \text{Eq. 2-70}$$

$$F_{p,i} = F_{p,i+1} + \frac{dF_p}{dx} \Delta x \qquad \text{Eq. 2-71a}$$

$$\int_{A_i} \rho \, g \, (h_i - z) \cos\Theta \, dA = \int_{A_{i+1}} \rho \, g \, (h_{i+1} - z) \cos\Theta \, dA + \frac{d\int_A \rho \, g \, (h-z) \cos\Theta \, dA}{dx} \Delta x \qquad \text{Eq. 2-71b}$$

If we substitute Eq. 2-71b in Eq. 2-68, we obtain the following expression:

$$\begin{aligned} 0 = &+\int_{x_i}^{x_{i+1}} \rho \ \frac{d\left(\alpha' v^2 A\right)}{dx} \ dx \ - \int_{x_i}^{x_{i+1}} \rho \cos\varphi \ \ v_e q_e dx - \frac{d\int_A \rho g\left(h-z\right)\cos\Theta dA}{dx} \ \Delta x \\ &- \int_{x_i}^{x_{i+1}} \int_0^h \rho g(h-z)\cos\Theta\left(\frac{db(z)}{dx}\right) \ dz \ dx + \int_{x_i}^{x_{i+1}} \rho \ \ g \ \ I_R A\left(x\right) \ dx \\ &- \int_{x_i}^{x_{i+1}} \rho \ \ g \ \ \tan\Theta\cos\Theta \ \ A\left(x\right) \ \ dx \end{aligned} \qquad \text{Eq. 2-72}$$

The change of the hydrostatic pressure force is calculated with application of the Leibniz theorem:

$$\begin{aligned} \frac{dF_p}{dx}\Delta x &= \frac{d}{dx}\left[\int_A \rho\, g\left(h-z\right)\cos\Theta\, dA\right]\Delta x \\ &= \left[\frac{d}{dx}\int_0^h \rho\, g\left(h-z\right)\cos\Theta\, b\left(z\right)dz\right]\Delta x \\ &= \rho\, g\cos\Theta\left[\left(\frac{dh\left(x\right)}{dx}\right)\int_0^h b\left(z\right)dz + \int_0^h \left(h-z\right)\left(\frac{db\left(z\right)}{dx}\right)dz\right]\Delta x \end{aligned} \qquad \text{Eq. 2-73}$$

with: $\int_0^h b\left(z\right)\, dz = A\left(x\right)$ we get:

$$\begin{aligned} \frac{dF_P}{dx} \ \Delta x &= \left[\rho g\cos\Theta \ \ A\left(x\right)\left(\frac{dh\left(x\right)}{dx}\right) + \int_0^h \rho g\left(h-z\right)\cos\Theta\left(\frac{db\left(z\right)}{dx}\right) \ dz\right] \ \Delta x \\ &= -\int_{x_i}^{x_{i+1}} \rho g\cos\Theta \ \ A\left(x\right)\left(\frac{dh\left(x\right)}{dx}\right) \ dx \ - \int_{x_i}^{x_{i+1}}\int_0^h \rho g\left(h-z\right)\cos\Theta\left(\frac{db\left(z\right)}{dx}\right) \ dz \ \ dx \end{aligned} \qquad \text{Eq. 2-74}$$

The last term corresponds exactly to the pressure force at the channel boundaries, but with opposite sign, so that these two terms cancel each other out in Eq. 2-72.

With regard to Eq. 2-74 in Eq. 2-72 and elimination of the integrals, the following differential equation is obtained:

$$\begin{aligned} 0 = &-\rho\frac{d\left(\alpha' v^2 A\right)}{dx} + \rho\cos\varphi \ v_e \, q_e - \rho\, g\cos\Theta\, A\left(x\right)\left(\frac{dh\left(x\right)}{dx}\right) - \rho\, g\, I_R \ A\left(x\right) \\ &+ \rho\, g\tan\Theta\cos\Theta\, A\left(x\right) \end{aligned} \qquad \text{Eq. 2-75}$$

This equation can be simplified to:

$$\frac{1}{g\,A}\frac{d\left(\alpha' \ Q^2 / A\right)}{dx} + \cos\Theta \ \frac{dz_{WSP}}{dx} + I_R - \cos\varphi\frac{v_e \ q_e}{g\,A} = 0 \qquad \text{Eq. 2-76}$$

Applying the product and quotient rule to the first term finally yields the following general differential form of the momentum balance equation:

$$\frac{1}{g}\left(\frac{Q^2}{A^2}\frac{d\alpha'}{dx}+\alpha'\frac{Q^2}{A^3}\frac{dA}{dx}\right)+\cos\Theta\frac{d\,z_{WSP}}{dx}+I_R+\left(\frac{2\,\alpha' Q}{A}-\cos\varphi\, v_e\right)\frac{q_e}{g\,A}=0 \qquad \text{Eq. 2-77}$$

Integrating over the control volume we get:

$$\frac{1}{g}\int_{x_i}^{x_{i+1}}\left(\frac{Q^2}{A^2}\frac{d\alpha'}{dx}-\alpha'\frac{Q^2}{A^3}\frac{dA}{dx}\right)dx+\int_{x_i}^{x_{i+1}}\cos\Theta\frac{d\,z_{WSP}}{dx}\,dx+\int_{x_i}^{x_{i+1}}I_R\,dx$$
$$+\int_{x_i}^{x_{i+1}}\left(\frac{2\,\alpha' Q}{A}-\cos\varphi\, v_e\right)\frac{q_e}{g\,A}\,dx=0 \qquad \text{Eq. 2-78}$$

Applying the trapezoidal formula that can generally be stated as follows:

$$\int_{x_1}^{x_2} f(x)\,dx=\frac{1}{2}(x_2-x_1)\left(f(x_2)+f(x_1)\right) \qquad \text{Eq. 2-79}$$

And assuming a linear variation of $A(x)$ and $\alpha'(x)$ along Δx, i.e.:

$$\frac{d\,A(x)}{dx}=\frac{A(x_2)-A(x_1)}{x_2-x_1} \quad \wedge \quad \frac{d\,\alpha'(x)}{dx}=\frac{\alpha'(x_2)-\alpha'(x_1)}{x_2-x_1} \qquad \text{Eq. 2-80}$$

the integral formula Eq. 2-78 can be solved as follows:

$$\frac{1}{2g}\left[(\alpha'_{i+1}-\alpha'_i)\left(\frac{Q_{i+1}^2}{A_{i+1}^2}+\frac{Q_i^2}{A_i^2}\right)-(A_{i+1}-A_i)\left(\alpha'_{i+1}\frac{Q_{i+1}^2}{A_{i+1}^3}+\alpha'_i\frac{Q_i^2}{A_i^3}\right)\right]$$
$$+\cos\Theta\left(z_{WSP,i+1}-z_{WSP,i}\right)+\frac{x_{i+1}-x_i}{2}\left(I_{R,i+1}+I_{R,i}\right)+\int_{x_i}^{x_{i+1}}\left(\frac{2\,\alpha' Q}{A}-\cos\varphi\, v_e\right)\frac{q_e}{g\,A}\,dx=0 \qquad \text{Eq. 2-81}$$

Finally, local momentum losses F_E caused by constructs like pillars, abutments, etc. can be accounted for in this equation making use of the resistance relationship:

$$F_E=\frac{1}{2}\rho\, c_w\, A_E\, v_a^2 \qquad \text{Eq. 2-82}$$

with: F_E = force that is absorbed by the constructs.
A_E = Area of the constructs that are subject to flow.
v_a = Flow velocity.
c_W = Drag coefficient of the construct.

With this improvement the so-called working equation of water line calculation based on momentum balance is finally obtained:

$$
\begin{aligned}
&\frac{1}{2g}\left[\left(\alpha'_{i+1}-\alpha'_{i}\right)\left(\frac{Q_{i+1}^2}{A_{i+1}^2}+\frac{Q_i^2}{A_i^2}\right)-\left(A_{i+1}-A_i\right)\left(\alpha'_{i+1}\frac{Q_{i+1}^2}{A_{i+1}^3}+\alpha'_i\frac{Q_i^2}{A_i^3}\right)\right] \\
&+\cos\Theta\left(z_{WSP,i+1}-z_{WSP,i}\right)+\frac{x_{i+1}-x_i}{2}\left(I_{R,i+1}+I_{R,i}\right)+\int_{x_i}^{x_{i+1}}\left(\frac{2\alpha' Q}{A}-\cos\varphi\, v_e\right)\frac{q_e}{g\,A}\,dx \\
&+\sum_{j=1}^{n}\frac{1}{2g}c_{w,j}\,\frac{A_{E,j}}{A_j}\,v_{a,j}^2=0
\end{aligned}
\qquad \text{Eq. 2-83}
$$

with: n = Number of constructs in section Δx
A_j = Cross sectional area at construct j

2.1.3 NUMERICAL ANALYSIS FOR SOLUTIONS OF THE EQUATION OF WORK

Mathematically seen the solution of Eq. 2-83 is a typical initial value problem. Based on a well-known water level at the profile i, the water level at the profile i +1 can be calculated. In relation to the water level the Eq. 2-83 is non-linear, so that it can not be solved directly and therefore an iterative solution algorithm is required. The best-known engineering methods to solve this equation is the standard-step method of CHOW (1959). Here initially the water level is estimated (Predictor), then the water level is adjusted by the calculation (Corrector) until Predictor and Corrector only differ with a negligible small error. Better convergence is obtained, however, with differentiated procedures, such as the method of continuous interval generation (NAUDASCHER, 1992) or the Newton method.

The calculation can be done both with and against the flow direction. At shooting flow it is always made in the direction of flow, as disturbances at the profile continue only in the direction of tail water. At steady flow calculation can be basically made in both directions. However, a calculation in the direction of flow is unfavourable, as in this case the estimated amount of hydrostatic water level z_{sp} at the profile i at the beginning still has effects on the last profile of the tail water in the calculated section. In contrast upstream, depending on the backwater situation, the water level at the profile i does not affect the flow situations at a certain profile in the upstream any more.

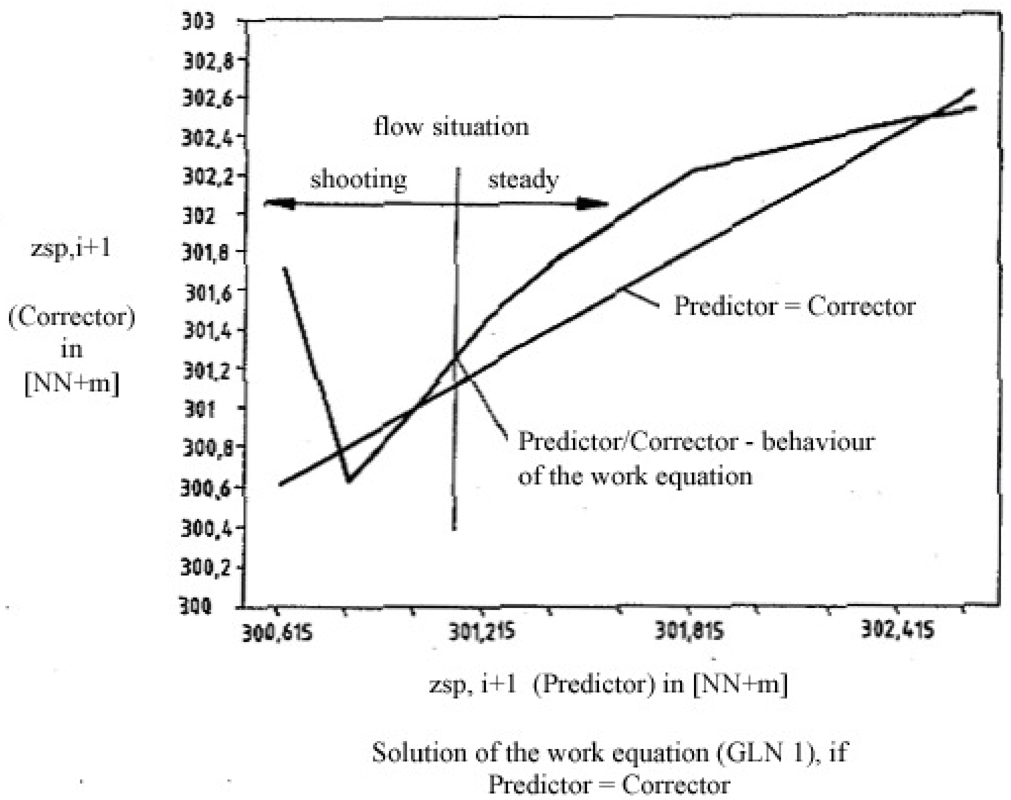

Figure 2-10: Possible solutions of the basic equations (Eq. 2-83)

How SEUS / USLU (1974) pointed out, there are for a maximum 3 solutions for the basic equation, of which the first two solutions are situated in the shooting part and the third solution in the steady part (Figure 2-10). Thereof the first solution is physically nonsensical. Even the second, shooting solution at the profile i +1 has to be discarded from physical considerations, if at the tail water profile i prevails steady flow.

If there are only 2 solutions in a calculation, contrary to the direction of flow, then this points to a flow transition between profile i and i +1. In this case, the calculation on the profile i +1 is not allowed to be continued with one of these solutions, because if a flow transition occurs the upstream water level is no longer dependent from the tail water profile. The causes for such a flow transition can be:

a) A vertical bend occurs between the profiles i and i +1 with a significant increase of the inclination at the profile i +1.
b) At the profile i +1 exists a bed drop or a cross-sectional constriction.

In the case of a) the shooting flow will be continued in the upstream direction. For a correct calculation of the water level diagram, the calculation at the profile i has to be cancelled and has to be continued upstream at a cross-profile, where the water level or the energy gradient is known, in the direction of flow to the profile i. It is important to consider whether a new flow transition occurs from shooting to steady flow. This condition occurs if:

$$Z_{E,min,i} > Z_{E,i+1} - h_r$$

In the case b) shooting flow occurs only in the profile i +1, since the flow transition is caused by a local discontinuity. In this case, the critical water depth at the profile i +1 is not solved with the basic equation (Eq. 2-83), but by applying the extremal doctrine. Thereafter the critical water depth is calculated, for example on a rectangular profile as:

$$z_{sp,i+1} = z_{So,i+1} + \frac{2}{3} h_{E,\min} \qquad \text{Eq. 2-84}$$

It may also be situations in which the basic equation provides no solution for the upstream profile. This can usually be ascribed to a too large distance between the profiles with a vertical bend with a change from steeper to flatter bed gradient (seen in direction of flow), as in this case, the integration of the friction loss through the trapezoidal formula leads to an underestimation of the friction loss. In most cases this problem can be solved through interconnecting other profiles. Approximately, also in this situation stationary steady flow can be adopted at the upstream profile, but at least the critical water depth. The water level at the profile i +1 can be solved by putting the energy gradient I_E on the same level with the bed gradient I_{SO} between the profiles i +1 and i.