

3 NUMERICAL METHODS

3.1 FINITE ELEMENT METHOD (FEM)

3.1.1 INTRODUCTION

The Finite Element Method is a numerical method for solving differential equations and integrals, and it is primarily used for problem solving in applied engineering and science. The Finite Element Method is a generalization of the well-established variation approach, which is based on the idea that the solution u of a differential equation can be approximated by a linear combination of the parameter c_j and appropriate functions Φ_j as shown in Eq. 3-1 [Reddy *et al.*, 1994].

$$u \approx u_N \equiv \sum_{j=1}^N c_j \Phi_j(x) + \Phi_0 \quad \text{Eq. 3-1}$$

In Eq. 3-1, u stands for the exact and u_N for the approximated (with FEM) solution. Thus, u stands for the vector of the unknown, which is in this case the water level and current velocities.

The parameters c_j are generally determined with the help of a weighted integral, so that they are a solution of the differential equation of the problem. When selecting the functions Φ_j , also called approximation or interpolation functions, it is important that they meet the boundary conditions. There are different methods for the variation approach like for example the Rayleigh-Ritz method or the method of weighted residuals, while the latter can further be distinguished into the Galerkin method, the least square method, and so on. The mentioned methods mainly differ in the choice of the weighting function ψ and the approximation function Φ . The **Galerkin method** for example, which is used in chapter 3.1.8 in the finite element form of the depth-averaged shallow water equation, requires the weighting function to be equal to the approximation function ($\psi = \Phi$). This is the reason why the introduction emphasizes this method.

More detailed explanations about the principles of the variation approach and the corresponding methods can be found in [Bathe, 1996] and [Reddy, 1993].

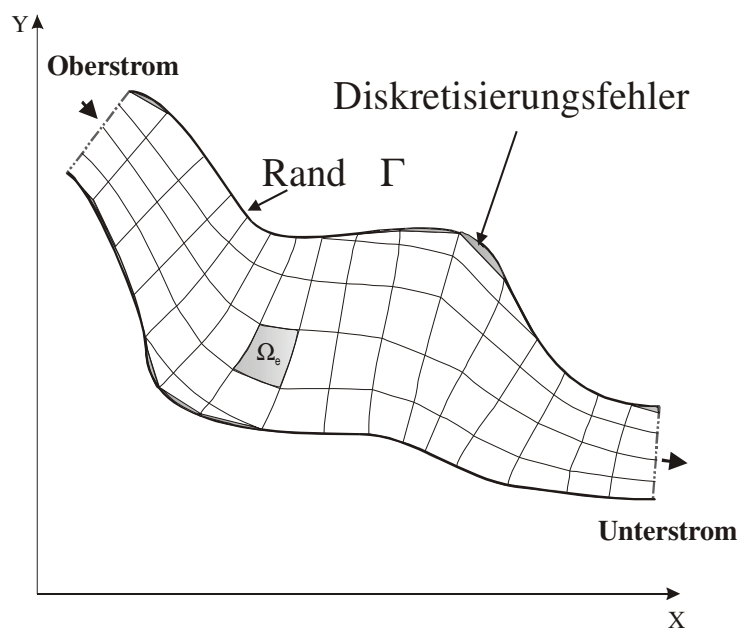


Figure 3-1: Finite element discretisation at the example of a river floodplain.

One of the main disadvantages of the variation approach becomes clear when applying it to complex geometric problems – and most problems in practice are of this type. It is difficult to find an appropriate approximation functions here, because these functions depend on the area geometry, among other dependencies, which will be shown in chapter 3.1.4. Now the finite element principle comes into play. The solution area is divided into a finite number of small sub areas (finite elements), as can be seen in Fig. 3-1. This is also called a finite element discretization of the model area. The geometry of the sub areas is selected to be simple (triangular or rectangular), so that the approximation functions that are necessary for the variation approach can be generated systematically and after a certain pattern. The solving of the differential equation is then related to vertices of the finite elements; if square approximation functions are used, additional vertices are generated on the midpoints of edges and maybe in the centre of elements.

In the following a short introduction to the finite element method shall be given. It does not claim to be a complete and exhaustive explanation of the method however. A number of excellent standard works like for example [Bathe, 1996] and [Reddy, 1993] exist that can be suggested for further reading. The introduction is rather meant to be a repetition of known material that is supposed to emphasize the main characteristics of the finite element method and that hopefully has an “oh-that-is-how-it-was”-effect for our advanced students. The access to the finite element form of the depth-averaged shallow water equation will probably be easier with this jump start.

The problem analysis prior to the finite element method is divided into the following steps according to [Reddy et al., 1994]:

Theory of the Finite Element Method	Application in practice
(1) Discretization of the model area into a finite number of finite elements.	Development of a FE-net with river channel and floodplain under consideration of the topographic and the classification of roughness
(2) Derivation of the weighted residual integral or the weak form of the differential equation to investigate and implement the finite element approach for the respective equations.	(<i>internal program processes</i>)
(3) Combining the elements in a global, algebraic system of equations.	(<i>internal program processes</i>)
(4a) Applying boundary and initial conditions to the system of equations	Involvement and approach of all hydrodynamic boundary conditions (water level-, runoff-hydrograph, W-Q-relations). Choice of a suitable mathematical start solution (lake solution or completely dry)
(4b) Merging area-specific parameters (roughness, eddy viscosity/ turbulence viscosity, fall dry, re-wetting)	Election and levying of all necessary hydraulic parameters. Adjustment within the calibration and/or sensitivity analysis
(5) Solving of the system of equations.	(<i>internal program processes</i>) <i>Monitoring of intermediate results</i>
(6) Verification, visualisation and validation of the solution	Monitoring, evaluation, visualisation (!) and review of the solution. Visualisation of complex results.

Tabelle 3-1: Work process of a problem analysis using the Finite Element Method

The monitoring and evaluation of erroneous or as well correct results is only possible with the understanding and the knowledge of the “internal program” processes. Only than, if applicable, possible sources of error and wrong assumptions in the modelling can be found.

The points (1)-(3) will be described in chapters 3.1.2 to 3.1.5 by hand of a simple example, while the points (4) and (5) will only be mentioned at the side and in relation to the program RMA2 (respectively RMA10s).

3.1.2 METHOD OF WEIGHTED RESIDUALS → FINITE ELEMENT FORM

This section does not refer to the FEM in general, but only to the specific method of weighted residuals with application of the Galerkin method.

First we have a look at the general differential equation Eq. 3-2, for which we want to find the solution u .

$$D(u) = q \quad \text{Eq. 3-2}$$

D is a linear operator here, in this case a differential operator, and q is some kind of outer load. If we substitute the approximation u_N from Eq. 3-1 into Eq. 3-2, the initial equation is not exactly satisfied anymore, and a remainder, also called residual, is generated.

$$R \equiv D(u_N) - q = D\left(\sum_{j=1}^N c_j \Phi_j + \Phi_0\right) - q = D\left(\sum_{j=1}^N c_j \Phi_j\right) + D(\Phi_0) - q \neq 0 \quad \text{Eq. 3-3}$$

Assuming u to be a function of only x and y (i.e. a two-dimensional, steady problem), the residual R is also a function of x and y , but also of c_j . With help of the method of weighted residuals, the parameters c_j are chosen so that the residual R approaches zero. The weighted integral below has to be solved.

$$\int_{\Omega} \psi_i(x, y) \cdot R(x, y, c_j) dx dy = 0 \quad (i = 1, 2, \dots, N) \quad \text{Eq. 3-4}$$

Integration is over the area Ω (two-dimensional area) and ψ_i are the weighting functions, that are principally different from the approximation functions Φ_j . Only for the Galerkin method ψ_i and Φ_j are set equal.

Eq. 3-1 to 3-4 are strictly speaking not a finite element formulation. Eq. 3-1 has to be modified first:

$$u(x, y) \approx u^e(x, y) = \sum_{j=1}^n u_j^e \psi_j^e(x, y) \quad \text{Eq. 3-5}$$

where $u^e(x, y)$ is an approximation of the solution $u(x, y)$ over the element Ω_e with the vertex count n ; u_j^e is the value of the function of $u^e(x, y)$ at the vertex j of the element and $\psi_j^e(x, y)$ is the approximation function for the element. Note that the definition $\psi = \Phi$ has already been made in Eq. 3-5 according to the Galerkin method.

If Eq. 3-5 is substituted into 3-4, we obtain the following general expression for the finite element form according to the method of weighted residuals:

$$\int_{\Omega_e} \psi_i^e(x, y) \cdot \left\{ D\left(\sum_{j=1}^n u_j^e \psi_j^e(x, y)\right) + D(\psi_0^e) - q \right\} dx dy = 0 \quad \text{Eq. 3-6}$$

3.1.3 THE „WEAK FORM“

The weak form of a differential equation is a weighted integral similar as in Eq. 3-6 and it satisfies as well the differential equation it is based on as the natural boundary conditions it is connected with. There is a weak form of each differential equation of the order $m \geq 2$, because only then the order of the derivative of the variable can be reduced by partial differentiation (e.g. Gauss theorem). Metaphorically speaking, the differentiation of the variable is transferred to the weighting function. At the same time, the natural boundary conditions are integrated into the weighted integral expression by means of the boundary integrals.

A simple example shall clarify the derivation of the weak form. The goal is the steady temperature distribution $T(x,y)$ in a two-dimensional isotropic medium Ω with the boundary Γ . The following equation (Poisson equation) can be stated to describe this problem:

$$-\left\{ \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) \right\} = Q \quad \text{in } \Omega \quad \text{Eq. 3-7}$$

where k is the heat conduction coefficient in direction of the x and y -axis and $Q(x,y)$ is the provided heat generation per volume unit (source term).

The following boundary conditions are defined prior to solving the differential equation Eq. 3-7:

$$T = \hat{T}(s) \quad \text{on } \Gamma_T \quad \text{Eq. 3-8}$$

$$\left(k \frac{\partial T}{\partial x} n_x + k \frac{\partial T}{\partial y} n_y \right) + q_c = q(s) \quad \text{on } \Gamma_q \quad \text{Eq. 3-9}$$

The boundaries Γ_T and Γ_q do not overlap and $\Gamma = \Gamma_T \cup \Gamma_q$, s is the coordinate along the boundary line, (n_x, n_y) is the unit vector normal to the boundary and q_c is the convective heat conduction defined by:

$$q_c = h_c(s, T)(T - T_c) \quad \text{Eq. 3-10}$$

and h_c is the convective heat conduction coefficient and T_c is the reference temperature.

The following finite element approximation is introduced for the variable T (temperature):

$$T(x, y) \approx T^e(x, y) = \sum_{j=1}^n T_j^e \psi_j^e(x, y) \quad \text{Eq. 3-11}$$

(cmp. Eq. 3-5)

In analogy to Eq. 3-6 an integral for the weighted residuals will be derived as one of the first steps for the derivation of the weak form.

$$0 = \int_{\Omega^e} \psi \left\{ -\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) - \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) - Q \right\} dx dy \quad \text{Eq. 3-12}$$

Using partial differentiation:

$$\frac{\partial}{\partial x}(g h) = h \frac{\partial g}{\partial x} + g \frac{\partial h}{\partial x} \quad \Leftrightarrow \quad -g \frac{\partial h}{\partial x} = h \frac{\partial g}{\partial x} - \frac{\partial}{\partial x}(g h) \quad \text{Eq. 3-13}$$

and the gradient theorem:

$$\int_{\Omega^e} \frac{\partial}{\partial x}(g h) dx dy = \oint_{\Gamma^e} (g h) n_x dx \quad \text{Eq. 3-14}$$

as well as the substitution:

$$h = k \frac{\partial T}{\partial x} \quad \text{and} \quad g = \psi \quad \text{Eq. 3-15}$$

Eq. 3-12 can be transformed to:

$$0 = \int_{\Omega^e} \left(k \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial x} + k \frac{\partial \psi}{\partial y} \frac{\partial T}{\partial y} - \psi Q \right) dx dy - \oint_{\Gamma^e} \psi q_n ds \quad \text{Eq. 3-16}$$

where q_n is the heat flux orthogonal to the boundary, which can also be written as:

$$q_n = k \frac{\partial T}{\partial x} n_x + k \frac{\partial T}{\partial y} n_y \quad \text{Eq. 3-17}$$

The variables n_x and n_y are the components of the normal vector along the boundary.

The integral expression from Eq. 3-16 is called “weak form” of Eq. 3-7. The reader should now recall the first section of this chapter (... flipping pages back might be necessary). If we have a look at Eq. 3-16, we notice that the order of the differential referring to the variable T is not m=2 but m=1 in the integral (reduction), but at the same time a differential for ψ has been added (transport from T to ψ). Additionally, the natural boundary condition from Eq. 3-9 has been integrated into the weak form by means of the boundary integral.

In order to transfer the weak form into a finite element form in the next step, the finite element approximation for T from Eq. 3-11 is substituted into Eq. 3-16 and we obtain:

$$0 = \int_{\Omega^e} \left\{ \frac{\partial \psi_i}{\partial x} \left(k \sum_{j=1}^n T_j^e \frac{\partial \psi_j^e}{\partial x} \right) + \frac{\partial \psi_i}{\partial y} \left(k \sum_{j=1}^n T_j^e \frac{\partial \psi_j^e}{\partial y} \right) - \psi_i Q \right\} dx dy - \oint_{\Gamma^e} \psi_i q_n ds \quad \text{Eq. 3-18}$$

After a few modifications, this equation can be written in matrix form

$$[K^e] \{T^e\} = \{Q^e\} + \{q^e\} \quad \text{Eq. 3-19}$$

with

$$K_{ij}^e = \int_{\Omega^e} \left(k \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} + k \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} \right) dx dy \quad \text{Eq. 3-20}$$

$$Q_i^e = \int_{\Omega^e} Q \psi_i^e dx dy, \quad q_i^e = \oint_{\Gamma^e} q_n \psi_i^e ds \quad \text{Eq. 3-21}$$

The matrix $[K^e]$ is the so-called coefficient matrix, also called stiffness matrix in statics. Eq. 3-19 above is quasi a first important milestone in our efforts to understand the finite element method, that is to say the (“weak”) finite element form of Eq. 3-7. In order to derive the weak form of the depth-averaged shallow water equation in chapter 3.1.8, we could actually make a break at this point, but unfortunately one who has the inconvenience of having to make practical use of the finite element method is confronted with a number of mathematical-numerical problems. The approaches for solving these problems have an influence on the solution of course. When evaluating the result, the user (for example of a commercial software) not only has to be able to judge the preconditions and the validity of the basic mathematical equation, but also the mathematical-numerical methods that are used. For this reason, additional questions will be discussed at this point like element connections and setting up the global system of equations, properties of and requirements for the approximation functions, numerical integration and time-discretization for unsteady problems.

3.1.4 THE APPROXIMATION FUNCTIONS

For the finite element approximation $T^e(x,y)$ of $T(x,y)$ to converge against the true solution for the element \mathcal{Q}^e , the approximation functions have to match a few criteria:

- (1) They have to be continuous according to the weak form of the problem, meaning that all terms of the weak form have to be unequal to zero. Consequently the approximation functions are sufficiently differentiable.
- (2) The polynomial that make up the approximation function have to be complete, i.e. all terms of the polynomial starting with the constant to the term of the highest degree have to be explicit (not equal to zero).
- (3) The individual terms of the polynomials have to be linearly independent.

In the following a simple example shall show the determination of linear approximation functions for a rectangular element. A simple rectangle is chosen for the elements because the approximation functions are functions of the geometry of the element. The sides of the elements in direction of the x and y-axis have the length a and b respectively. Additionally, a local coordinate system is defined that has its origin in the vertex 1 (see Fig. 3-2). The introduction of the local coordinate system is a first taste of the coordinate transformation and the so-called master element that will be explained in chapter 3.1.6.

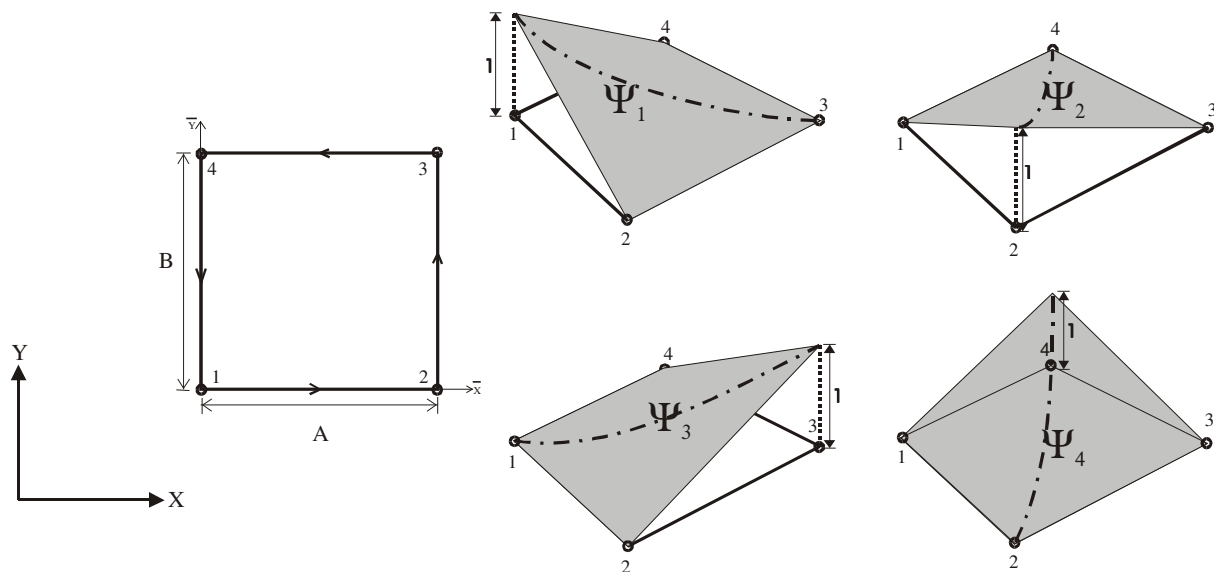


Figure 3-2: Rectangular element with linear approximation functions.

The general polynomial for the approximation functions expressed with global coordinates is:

$$T^e(x, y) = c_1 + c_2 x + c_3 y + c_4 xy \quad \text{Eq. 3-22}$$

and in the local coordinate system:

$$T^e(\bar{x}, \bar{y}) = c_1 + c_2 \bar{x} + c_3 \bar{y} + c_4 \bar{x} \bar{y} \quad \text{Eq. 3-23}$$

If the function values are substituted according to Fig. 3-2 into Eq. 3-23, the following relations are obtained for the constants:

$$\begin{aligned} T_1 &= T(0,0) = c_1 \\ T_2 &= T(a,0) = c_1 + c_2 a \\ T_3 &= T(a,b) = c_1 + c_2 a + c_3 b + c_4 ab \\ T_4 &= T(0,b) = c_1 + c_3 b \end{aligned}$$

Eq. 3-24

$$\begin{aligned} \Leftrightarrow c_1 &= T_1 \\ c_2 &= \frac{T_2 - T_1}{a} \\ c_3 &= \frac{T_4 - T_1}{b} \\ c_4 &= \frac{T_3 - T_4 + T_1 - T_2}{ab} \end{aligned}$$

Substitution of the relations for c_1 , c_2 , and so on into Eq. 3-23 yields the following expressions for the approximation functions:

$$\begin{aligned} T(\bar{x}, \bar{y}) &= T_1 \left(1 - \frac{\bar{x}}{a} - \frac{\bar{y}}{b} + \frac{\bar{x}\bar{y}}{ab} \right) + T_2 \left(\frac{\bar{x}}{a} - \frac{\bar{x}\bar{y}}{ab} \right) + T_3 \frac{\bar{x}\bar{y}}{ab} + T_4 \left(\frac{\bar{x}}{b} - \frac{\bar{x}\bar{y}}{ab} \right) \\ &= T_1 \psi_1 + T_2 \psi_2 + T_3 \psi_3 + T_4 \psi_4 = \sum_{i=1}^4 T_i \psi_i \end{aligned}$$

Eq. 3-25

$$\Leftrightarrow \psi_1 = \left(1 - \frac{\bar{x}}{a} \right) \left(1 - \frac{\bar{y}}{b} \right), \quad \psi_2 = \frac{\bar{x}}{a} \left(1 - \frac{\bar{y}}{b} \right), \quad \psi_3 = \frac{\bar{x}\bar{y}}{ab}, \quad \psi_4 = \left(1 - \frac{\bar{x}}{a} \right) \frac{\bar{y}}{b}$$

In matrix form:

$$\begin{Bmatrix} 1 - \frac{\bar{x}}{a} \\ \frac{\bar{x}}{a} \end{Bmatrix} \begin{Bmatrix} 1 - \frac{\bar{y}}{b} & \frac{\bar{y}}{b} \end{Bmatrix} = \begin{bmatrix} \psi_1 & \psi_4 \\ \psi_2 & \psi_3 \end{bmatrix}$$

Eq. 3-26

To be able to inspect the properties of these approximation functions, we have another look at Fig. 3-2, which contains the function values for ψ_i at the element vertices:

$$\begin{aligned} \psi_i(\bar{x}_i, \bar{y}_i) &= 0 \quad (i=2,3,4) \\ \psi_1(\bar{x}_1, \bar{y}_1) &= 1 \end{aligned}$$

Eq. 3-27.

This means that the value of the function ψ_i is one for vertex 1 and zero for every other vertex. The result for the functions ψ_2 , ψ_3 and ψ_4 is the same.

The approximation functions for selected master elements (s. Fig. 3-3 and 3-4) are given now as a preview of what is to come in chapter 3.1.6, i.e. the coordinate transformation and the introduction of the so-called master element. How the latter are motivated and derived can be found in respective

technical literature. The choice of the approximation functions is based on their use in the finite element program RMA2. Some of these approximation functions will serve as examples too. In general, the difference between approximation functions is in their order, i.e. they are linear, square or of a higher order, and furthermore they are different in the applied element shape (geometry). All of these factors determine the necessary sampling rate or vertex count that defines how the function distributes over the element.

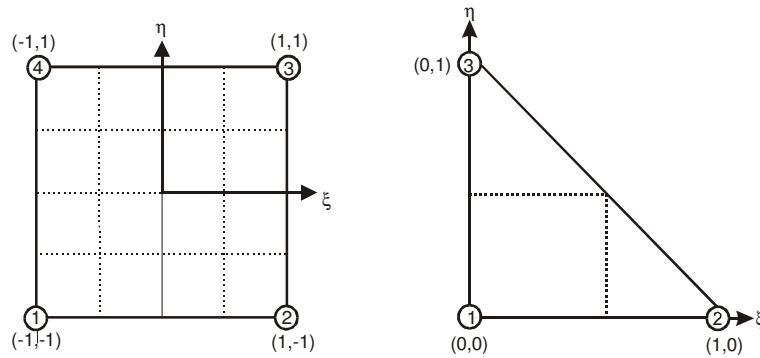


Figure 3-3: Rectangular and triangular Lagrangian element with linear approximation function, taken from [Stein, 1990].

$$\{\Psi^e\} = \frac{1}{4} \begin{Bmatrix} (1-\xi)(1-\eta) \\ (1+\xi)(1-\eta) \\ (1+\xi)(1+\eta) \\ (1-\xi)(1+\eta) \end{Bmatrix} \quad \{\Psi^e\} = \begin{Bmatrix} (1-\xi-\eta) \\ \xi \\ \eta \end{Bmatrix} \quad \text{Eq. 3-28}$$

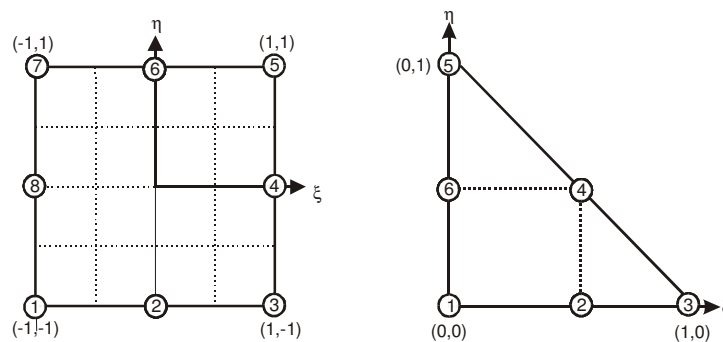


Figure 3-4: Rectangular and triangular Lagrangian element with square approximation function, taken from [Stein, 1990].

$$\{\Psi^e\} = \frac{1}{4} \begin{Bmatrix} (1-\xi)(1-\eta)(-\xi-\eta-1) \\ 2(1-\xi^2)(1-\eta) \\ (1+\xi)(1-\eta)(\xi-\eta-1) \\ 2(1+\xi)(1-\eta^2) \\ (1+\xi)(1+\eta)(\xi+\eta-1) \\ 2(1-\xi^2)(1+\eta) \\ (1-\xi)(1+\eta)(-\xi+\eta-1) \\ 2(1-\xi)(1-\eta^2) \end{Bmatrix} \quad \{\Psi^e\} = 2 \begin{Bmatrix} (1-\xi-\eta)(-\xi-\eta+\frac{1}{2}) \\ 2\xi(1-\xi-\eta) \\ \xi(\xi-\frac{1}{2}) \\ 2\xi\eta \\ \eta(\eta-\frac{1}{2}) \\ 2\eta(1-\xi-\eta) \end{Bmatrix} \quad \text{Eq. 3-29}$$

Equations 3-28 and 3-29 represent the approximation functions belonging to the figures 3-3 and 3-4 respectively.

3.1.5 VERTEX CONNECTIONS → GLOBAL SYSTEM OF EQUATIONS

Two factors have to be accounted for when elements are connected to form the global system of equations:

- (1) Continuity of the primary variables (here: temperature) across element bounds.
- (2) Mass balance of secondary variables (here: heat conduction) for the element.

The principles of connecting elements shall be shown with the help of an example taken from Fig. 3-5.

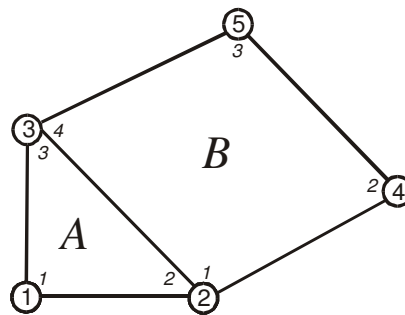


Figure 3-5: Local and global graph numbering, taken from [Reddy *et al.*, 1994].

The following relations between local and global values for the temperature T can be derived from the figure:

$$T_1^1 = T_1 \ ; \ T_2^1 = T_1^2 = T_2 \ ; \ T_3^1 = T_4^2 = T_3 \ ; \ T_2^2 = T_4 \ ; \ T_3^2 = T_5 \quad \text{Eq. 3-30}$$

Eq. 3-30 represents the first of the above requirements for the connecting of elements. If the vertex values of the adjacent vertices of elements 1 and 2 are identical, the continuity requirement at their common edge is met (... of course only if the approximation functions of element 1 and 2 are of the same order).

Now we look at the mass balance of the two elements. The following relationships hold for the heat conduction q_n^e e.g. from element 1 to element 2 (and vice versa):

$$(q_n^1)_{2-3} = (q_n^2)_{4-1} \quad \text{or} \quad (q_n^1)_{2-3} = -(q_n^2)_{1-4} \quad \text{Eq. 3-31}$$

The heat conduction q_n^e in Eq. 3-31 has to be positive pointing away from the edge of the element if we cycle counter-clockwise through the elements, according to the sign-convention. The finite element form contains the relationship expressed in Eq. 3-31 as a weighted integral and it can be written as:

$$\int_{h_{23}^1} q_n^1 \psi_2^1 ds = - \int_{h_{14}^2} q_n^2 \psi_1^2 ds \quad \text{Eq. 3-32}$$

$$\int_{h_{23}^1} q_n^1 \psi_3^1 ds = - \int_{h_{14}^2} q_n^2 \psi_4^2 ds$$

where h_{pq}^e is the length of the edge from vertex p to vertex q of element Ω^e .

The equations for the individual elements are now as follows:

Triangle, element 1

$$\begin{aligned} K_{11}^1 T_1^1 + K_{12}^1 T_2^1 + K_{13}^1 T_3^1 &= Q_1^1 + q_1^1 \\ K_{21}^1 T_1^1 + K_{22}^1 T_2^1 + K_{23}^1 T_3^1 &= Q_2^1 + q_2^1 \\ K_{31}^1 T_1^1 + K_{32}^1 T_2^1 + K_{33}^1 T_3^1 &= Q_3^1 + q_3^1 \end{aligned} \quad \text{Eq. 3-33}$$

Square, element 2

$$\begin{aligned} K_{11}^2 T_1^2 + K_{12}^2 T_2^2 + K_{13}^2 T_3^2 + K_{14}^2 T_4^2 &= Q_1^2 + q_1^2 \\ K_{21}^2 T_1^2 + K_{22}^2 T_2^2 + K_{23}^2 T_3^2 + K_{24}^2 T_4^2 &= Q_2^2 + q_2^2 \\ K_{31}^2 T_1^2 + K_{32}^2 T_2^2 + K_{33}^2 T_3^2 + K_{34}^2 T_4^2 &= Q_3^2 + q_3^2 \\ K_{41}^2 T_1^2 + K_{42}^2 T_2^2 + K_{43}^2 T_3^2 + K_{44}^2 T_4^2 &= Q_4^2 + q_4^2 \end{aligned} \quad \text{Eq. 3-34}$$

$K_{KnA,KnB}^{EL}$ stands here for the respective coefficient at the element “E1” related to the balance at the knot “KnA” (see solution vector) and the derivation over the unknown, which is here the temperature T at the knot “KnB”. Hence, the second, subscripted index of the knot number needs to correspond to the unknown which is here the temperature T_{KnB} .

In order to account for the balance of the primary variables according to Eq. 3-32, it is necessary to add the second equation of element 1 to the first equation of element 2, and the third equation of element 1 to the fourth equation of element 2. If concurrently the global vertex numbering is introduced for the primary variable T according to Eq. 3-30, the global system of equations for our two-element-graph can be clearly written in matrix form as:

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 & 0 & 0 \\ K_{21}^1 & (K_{22}^1 + K_{11}^2) & (K_{23}^1 + K_{14}^2) & K_{12}^2 & K_{13}^2 \\ K_{31}^1 & (K_{32}^1 + K_{41}^2) & (K_{33}^1 + K_{44}^2) & K_{42}^2 & K_{43}^2 \\ 0 & K_{21}^2 & K_{24}^2 & K_{22}^2 & K_{23}^2 \\ 0 & K_{31}^2 & K_{34}^2 & K_{32}^2 & K_{33}^2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_3^1 + Q_4^2 \\ Q_2^2 \\ Q_3^2 \end{bmatrix} + \begin{bmatrix} q_1^1 \\ q_2^1 + q_1^2 \\ q_3^1 + q_4^2 \\ q_2^2 \\ q_3^2 \end{bmatrix} \quad \text{Eq. 3-35}$$

3.1.6 NUMERICAL INTEGRATION

For purposes of error minimization during the finite element discretization of complex and non-uniform model areas, arbitrary triangles and squares are used as finite elements (see Fig. 3-1). Please note, that for triangles and also squares it is important to comply with specific rules including side lengths, angle and the ratio to the neighbour elements as well as with criteria such as the Delauny criterion.

These arbitrary element shapes make the calculation of the integrals more difficult, however, as for example for the coefficient matrix from Eq. 3-19 or 3-20. For this reason, a coordinate transformation is done, which maps the elements from a global coordinate system (x,y) to a so-called master element in a local coordinate system (ξ,η) . The coordinate transformation for a square is exemplarily displayed in Fig. 3-6. The master element is selected with thought of the numerical solving method used. If Gauss integration is used (this method is used in the program RMA2) for rectangular elements, for example, a square master element of edge length 2 having its midpoint in the origin of the coordinate system (ξ,η) and its edges parallel to the axes of the coordinate system is selected (see Fig. 3-3). The Gauss integration method is presumed to be used further on.

Although the coordinate transformation makes the wanted integral expression more complicated, the following simplifications are achieved by the introduction of a geometrically simple master element that is used in place of the actual elements:

- (1) The approximation functions can be obtained quite easily and they are the same for each element since they are always applied to the master element.
- (2) The calculation of the integrals for the element is simplified by a common, simple element geometry. Commonly available numerical integration methods can be used.

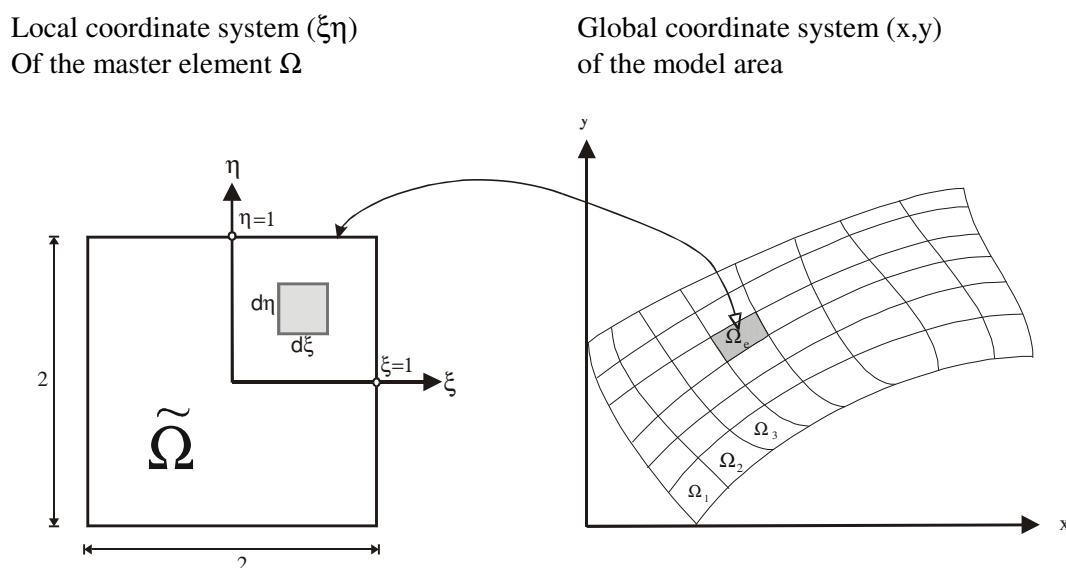


Figure 3-6: Transformation of a finite element graph consisting of rectangles to a square unit element (master element), taken from [Reddy et al., 1994].

It has to be emphasized that the coordinate transformation is done for calculation purposes of numerical integration only. The algebraic equations that have to be solved always refer to the wanted degrees of freedom (vertex values) in the global coordinate system.

The transformation of any element of the global coordinate system (x,y) to the master element of the local system (ξ,η) is described by the following, for example:

$$x = \sum_{j=1}^k x_j^e \psi_j^m(\xi, \eta) \quad , \quad y = \sum_{j=1}^k y_j^e \psi_j^m(\xi, \eta) \quad \text{Eq. 3-36}$$

where ψ_j^m is the approximation function for the master element. In order to make this transformation function clearer, we can assume a square master element with linear approximation functions, and the local coordinates satisfy the relation $-1 \leq (\xi, \eta) \leq 1$ (see Fig. 3-3). This is called a rectangular Lagrange element with vertex count $k=4$. The approximation functions for a master element of this type can be derived according to Eq. 3-28. If these linear approximation functions are used, the straight line $\xi=1$ (local coordinate system) can be describe by the following equations in the global coordinate system:

$$\begin{aligned} x(\eta) &= \sum_{i=1}^4 x_i \psi_i^m(\eta) = x_1 \cdot 0 + \frac{1}{2} x_2 (1-\eta) + \frac{1}{2} x_3 (1+\eta) + x_4 \cdot 0 \\ &= \frac{1}{2} (x_2 + x_3) + \frac{1}{2} (x_3 - x_2) \eta \\ y(\eta) &= \sum_{i=1}^4 y_i \psi_i^m(\eta) = \frac{1}{2} (y_2 + y_3) + \frac{1}{2} (y_3 - y_2) \eta \end{aligned} \quad \text{Eq. 3-37}$$

This means that x and y are well-defined linear functions of η meaning that the transformation maps the straight line $\xi=1$ also to a straight line in the global coordinate system.

Before we go into deeper detail of coordinate transformation, a few terms have to be explained. We recall the approximation for the solution of the wanted variable from Eq. 3-5 for this. It contains approximation functions just as in Eq. 3-36. If an approximation of the same order ($k=n$) is chosen for the transformation (geometry) and the wanted variable, the finite elements are called isoparametric. If the order of the approximation for the geometry is bigger than the order of the wanted variable ($k > n$), the elements are superparametric, and in the opposite case they are subparametric [Helmig, 1996], [Reddy et al., 1994]. Only isoparametric finite elements are used in the context of this chapter.

Our knowledge of coordinate transformation that we have now enables us to recognize that a different shape (e.g. triangle or rectangle) or order (e.g. linear or square) of the master element leads to different transformations and thus to different representations of the elements in the global coordinate system. It is important that the transformation does not introduce unwanted gaps or overlapping elements into the finite element graph. In order to set up appropriate requirements for the transformation, we have a look at the elements of the coefficient matrix from Eq. 3-20.

$$K_{ij}^e = \int_{\Omega^e} \left(k \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} + k \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} \right) dx dy \quad \text{Eq. 3-38}$$

The integrand that not only contains functions but also derivatives with respect to the global coordinates x and y , now has to be expressed in dependence of ξ and η . The transformation from Eq. 3-36 is applied to get the result. We first have to apply rules of partial differentiation however:

$$\frac{\partial \psi_i^e}{\partial \xi} = \frac{\partial \psi_i^e}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial \psi_i^e}{\partial y} \frac{\partial y}{\partial \xi}$$

$$\frac{\partial \psi_i^e}{\partial \eta} = \frac{\partial \psi_i^e}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial \psi_i^e}{\partial y} \frac{\partial y}{\partial \eta}$$

Eq. 3-39

$$\begin{pmatrix} \frac{\partial \psi_i^e}{\partial \xi} \\ \frac{\partial \psi_i^e}{\partial \eta} \end{pmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{pmatrix} \frac{\partial \psi_i^e}{\partial x} \\ \frac{\partial \psi_i^e}{\partial y} \end{pmatrix}$$

This equation reflects a connection between the derivatives of the approximation functions with respect to the global and local coordinates. The matrix in Eq. 3-39 is also called Jacobi transformation matrix. In order to obtain the needed expressions, we have to invert the equation above:

$$\begin{pmatrix} \frac{\partial \psi_i^e}{\partial x} \\ \frac{\partial \psi_i^e}{\partial y} \end{pmatrix} = [J]^{-1} \begin{pmatrix} \frac{\partial \psi_i^e}{\partial \xi} \\ \frac{\partial \psi_i^e}{\partial \eta} \end{pmatrix}$$

Eq. 3-40

Provided the transformation from Eq. 3-36 the individual components of the Jacobi matrix can be calculated as:

$$\frac{\partial x}{\partial \xi} = \sum_{j=1}^k x_j \frac{\partial \psi_j^m}{\partial \xi} \quad , \quad \frac{\partial y}{\partial \xi} = \sum_{j=1}^k y_j \frac{\partial \psi_j^m}{\partial \xi}$$

$$\frac{\partial x}{\partial \eta} = \sum_{j=1}^k x_j \frac{\partial \psi_j^m}{\partial \eta} \quad , \quad \frac{\partial y}{\partial \eta} = \sum_{j=1}^k y_j \frac{\partial \psi_j^m}{\partial \eta}$$

Eq. 3-41

In this case, the Jacobi matrix can be written as:

$$[J] = \begin{bmatrix} \frac{\partial \psi_1^m}{\partial \xi} & \frac{\partial \psi_2^m}{\partial \xi} & \dots & \frac{\partial \psi_n^m}{\partial \xi} \\ \frac{\partial \psi_1^m}{\partial \eta} & \frac{\partial \psi_2^m}{\partial \eta} & \dots & \frac{\partial \psi_n^m}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

Eq. 3-42

The inverse (reciprocal) Jacobi matrix $[J]^{-1}$ only exists if $[J]$ is non-singular, i.e. the determinant of the Jacobi matrix is unequal to zero for each point (ξ, η) of the master element [Zurmühl et al., 1984]:

$$\det[J] = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} \neq 0 \quad \text{Eq. 3-43}$$

Altogether it can be stated that the transformation has to be continuous, differentiable and invertible. At the same time it should be relatively simple, so that the Jacobi matrix can be calculated with little effort.

In order to be able to apply the integration completely to Eq. 3-38, we need a transformation of the area expression dA of the global element that corresponds to the transformation that is used:

$$dA \equiv dx \, dy = |\det[J]| \, d\xi \, d\eta \quad \text{Eq. 3-44}$$

Eq. 3-38 can now be written as follows on the basis of the transformation:

$$K_{ij}^e = \int_{\Omega^m} F_{ij}(\xi, \eta) \, d\xi \, d\eta \quad \text{Eq. 3-45}$$

The integrals in Eq. 3-45 that refer to the square master element can be calculated with the help of the Gauss quadrature method, so that the integral above can be calculated as follows for the two-dimensional case:

$$\int_{\Omega^m} F(\xi, \eta) \, d\xi \, d\eta = \int_{-1}^1 \int_{-1}^1 F(\xi, \eta) \, d\xi \, d\eta \approx \sum_{i=1}^K \sum_{j=1}^N F(\xi_i, \eta_j) W_i W_j \quad \text{Eq. 3-46}$$

where K and N are the number of Gauss quadrature points, (ξ_i, η_j) are the Gauss coordinates, and W_i and W_j are the values of the weighting functions.

More detailed information about the application of the Gauss quadrature method can be found in [Reddy, 1993] among others.

3.1.7 UNSTEADY PROBLEMS

In order to explain unsteady problems, the heat conductivity problem from chapter 3.1.3 can be reused as an example. This time the unsteady temperature distribution $T(x,y,t)$ in a two-dimensional isotropic medium Ω with the boundary Γ is wanted. The differential equation for this problem is:

$$\rho C \frac{\partial T}{\partial t} - \left\{ \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) \right\} = Q \quad \text{in } \Omega \quad \text{Eq. 3-47}$$

with the density ρ , the specific heat C , the heat conductivity coefficient k in direction of the x and y -axis (isotropic medium) and the heat generation per volume unit $Q(x,y,t)$ (source term). The boundary condition that is necessary to solve the differential equation is given by the initial solution at the time $t=0$.

$$T(x, y, 0) = T_0(x, y) \quad \text{Eq. 3-48}$$

To allow for time-dependent problems to be solved completely with the finite element method, special elements are necessary, that can be integrated as well spatially as with respect to time. This means a lot of additional work, although the finite element method itself is very time-consuming. The result is that the already complex solving strategy becomes even more abstract and more difficult to understand. That is why a so-called semi-discrete method is used in many applications of the finite element method (e.g. in RMA2). In this case it is assumed that it is possible to separate the dependency on time from the spatial variation. In the semi-discrete method just like for the steady problem, the weak form is derived, which in our case can be expressed as:

$$0 = \int_{\Omega^e} \left[\psi \left(\rho C \frac{\partial T}{\partial t} - Q \right) + \left(k \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial x} + k \frac{\partial \psi}{\partial y} \frac{\partial T}{\partial y} \right) \right] dx dy - \oint_{\Gamma^e} \psi q_n ds \quad \text{Eq. 3-49}$$

The equation above makes the assumptions of the Galerkin method, i.e. approximation function = weighting function. The terms of Eq. 3-49 depend on the time as opposed to the weak form of the steady problem. No partial differentiation is done with respect to time, and the weighting function is assumed to be a function of only x and y .

The separation of time-dependency of the problem and spatial variation that has already been mentioned can be stated as follows with respect to the approximate solution:

$$T(x, y, t) \cong T_N(x, y, t) \equiv \sum_{i=1}^n T_i^e(t) \psi_i^e(x, y) \quad \text{Eq. 3-50}$$

If this approximate solution is substituted into the weak form from Eq. 3-49 the following system of equations emerges (in matrix form):

$$[M^e] \left\{ \dot{T}^e \right\} + [K^e] \{T^e\} = \{Q^e\} + \{q^e\} \quad \text{Eq. 3-51}$$

with the components of the matrix being:

$$M_{ij}^e = \int_{\Omega^e} \rho C \psi_i^e \psi_j^e dx dy \quad , \quad K_{ij}^e = \int_{\Omega^e} \left(k \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} + k \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} \right) dx dy$$

Eq. 3-52

$$Q_i^e = \int_{\Omega^e} \psi_i^e Q(x, y, t) dx dy \quad , \quad q_i^e = \oint_{\Gamma^e} \psi_i^e q_n ds$$

\dot{T} stands for the derivative with respect to time, i.e. $\frac{\partial T}{\partial t}$.

The semi-discrete method for the calculation of the derivative with respect to time makes use of finite differences, i.e. discrete time intervals are used. Eq. 3-53 summarizes the most common approximations for time discretization, all of them being two-level methods, which means that they are based on values from two time-levels $n+1$ and n .

$$\{T\}_{n+1} = \{T\}_n + \Delta t \left[(1-\theta) \left\{ \dot{T} \right\}_n + \theta \left\{ \dot{T} \right\}_{n+1} \right] \quad \text{with : } 0 \leq \theta \leq 1$$

Eq. 3-53

The methods for different values for θ are called:

$$\begin{aligned} \theta = 0 \quad , \quad & \text{Forward – Euler – Methode} \\ \theta = \frac{1}{2} \quad , \quad & \text{Crank – Nicolson – Methode} \\ \theta = \frac{2}{3} \quad , \quad & \text{Galerkin – Methode} \\ \theta = 1 \quad , \quad & \text{Backward – Euler – Methode} \end{aligned}$$

Eq. 3-54

The forward-Euler method is also called explicit method because the calculation of every flow and every source term of the equation to be solved is based on the old time level t_n . The only variable at the time t_{n+1} is the variable in the vertex i . The variable is thus calculated without reference to any values of adjacent vertices (spatial derivatives), i.e. explicitly. Additional conditions have to be made for the time-step (time increment Δt) in order to achieve stable solutions with this method. These conditions are defined in dependency of the Peclet number. The time-step method used in RMA2 is an explicit method like as described above.

The backward-Euler method is an implicit method because the spatial derivatives, for example, are based only on the values of (adjacent) vertices of the current time level t_{n+1} . The implicit method has no restrictions concerning the width of time-steps and its effect on the stability of the calculation as opposed to the explicit method. On the other hand, it is an iterative method, which means a higher numerical effort for the individual time-steps.

More detailed explanations concerning the solving of unsteady problems can be found in [Ferziger et al., 1999] among others.

3.1.8 TWO-DIMENSIONAL, DEPTH-INTEGRATED FLOW

Before we proceed to the finite element form of our two-dimensional, depth-integrated flow problem (... which is also incompressible), we remember the basis equations, i.e. the shallow water equations, from chapter 2.2.1.3:

$$\frac{\partial h}{\partial t} + \frac{\partial h u_i}{\partial x_i} = 0$$

Eq. 3-55

$$h \frac{\partial u_i}{\partial t} + h u_j \frac{\partial u_i}{\partial x_j} + g h \frac{\partial}{\partial x_i} (h + z_0) = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[h \varepsilon_{ij} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] - \frac{\tau_{so,i}}{\rho}$$

The shallow water equation stated here already makes use of a number of simplifications for better readability. The influence of wind shear stress at the surface and the coriolis force, for example, have not been accounted for. For further simplification we have a closer look at the first term on the right side of the shallow water equation. The calculations will be done for the x-component:

$$\begin{aligned} & \frac{\partial}{\partial x} \left[h \varepsilon_{xx} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[h \varepsilon_{xy} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \\ &= \frac{\partial}{\partial x} \left[2 h \varepsilon_{xx} \left(\frac{\partial u}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[h \varepsilon_{xy} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \quad \text{with } (f g)' = f' g + f g' \quad \text{Eq. 3-56} \\ &= \underbrace{\frac{\partial u}{\partial x}}_{g1} \underbrace{\frac{\partial}{\partial x} (2 h \varepsilon_{xx})}_{f1'} + \underbrace{2 h \varepsilon_{xx}}_{f1} \underbrace{\frac{\partial^2 u}{\partial x^2}}_{g1'} + \underbrace{h \varepsilon_{xy}}_{f2} \underbrace{\left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial x} \right]}_{g2'} + \underbrace{\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right]}_{g2} \underbrace{\frac{\partial}{\partial y} (h \varepsilon_{xy})}_{f2'} \end{aligned}$$

Additionally, the following assumptions are made:

- (1) The derivatives $\left| \frac{\partial \varepsilon_{ij}}{\partial x_j} \right|, \left| \frac{\partial h}{\partial x_j} \right|$ are small \Rightarrow e.g. $\left| \frac{\partial u_i}{\partial x_j} \right| \cdot \left| \frac{\partial \varepsilon_{ij}}{\partial x_j} \right| \ll 1$, i.e. very small.
- (2) The turbulence is regarded as being isotropic, i.e. $\varepsilon_{xx} = \varepsilon_{xy} = \varepsilon_{yy}$.

With these assumptions, Eq. 3-56 can be written as:

$$\begin{aligned} & h \varepsilon_{xx} \frac{\partial^2 u}{\partial x^2} + h \varepsilon_{xy} \frac{\partial^2 u}{\partial y^2} + h \varepsilon_{xy} \frac{\partial^2 v}{\partial y \partial x} + h \varepsilon_{xx} \frac{\partial^2 u}{\partial x^2} + \underbrace{\frac{\partial u}{\partial x} \frac{\partial}{\partial x} (2 h \varepsilon_{xx})}_{\ll 1} \\ & \quad + \underbrace{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \frac{\partial}{\partial y} (h \varepsilon_{xy})}_{\ll 1} = 0 \end{aligned} \quad \text{Eq. 3-57}$$

Substituting into the shallow water equation (x-component) yields:

$$\begin{aligned} & h \frac{\partial u}{\partial t} + h u \frac{\partial u}{\partial x} + h v \frac{\partial u}{\partial y} + g \frac{\partial \left(\frac{h^2}{2} \right)}{\partial x} + g h \frac{\partial z_o}{\partial x} \\ & = \frac{1}{\rho} \left[h \varepsilon_{xx} \frac{\partial^2 u}{\partial x^2} + h \varepsilon_{yy} \frac{\partial^2 u}{\partial y^2} + h \varepsilon_{xy} \left[\frac{\partial}{\partial x} \left(\underbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}}_{=0} \right) \right] \right] - \frac{\tau_{so,x}}{\rho} \end{aligned} \quad \text{Eq. 3-58}$$

The equation can be further reduced using the continuity condition to:

$$h \frac{\partial u}{\partial t} + h u \frac{\partial u}{\partial x} + h v \frac{\partial u}{\partial y} + g \frac{\partial \left(\frac{h^2}{2} \right)}{\partial x} + g h \frac{\partial z_o}{\partial x} = \frac{h}{\rho} \left[\varepsilon_{xx} \frac{\partial^2 u}{\partial x^2} + \varepsilon_{yy} \frac{\partial^2 u}{\partial y^2} \right] - \frac{\tau_{so,x}}{\rho} \quad \text{Eq. 3-59}$$

This can be done analogously for the y-component, so that the basic system of equations (cmp. Eq. 3-55) can now be written as:

$$\frac{\partial h}{\partial t} + \frac{\partial h u_i}{\partial x_i} = 0 \quad \text{Eq. 3-60}$$

$$h \frac{\partial u_i}{\partial t} + h u_j \frac{\partial u_i}{\partial x_j} + g \frac{\partial \left(\frac{h^2}{2} \right)}{\partial x_i} + g h \frac{\partial z_o}{\partial x_i} = \frac{h}{\rho} \varepsilon_{ij} \frac{\partial^2 u_i}{\partial x_j^2} - \frac{\tau_{so,i}}{\rho}$$

These basis equations form the starting point for the finite element form of the two-dimensional, depth-integrated flow problem.

In order to deduce the finite element form, we follow a well-known “recipe” (... it should be studied so that it can be done standing on your head!) and deduce the weak form as the first step. For this purpose we set up weighted integrals first:

$$\int_{\Omega^e} Q f_1 \, dx \, dy \quad \text{and} \quad \int_{\Omega^e} W f_2 \, dx \, dy \quad \text{Eq. 3-61}$$

where Q and W are weighting functions and f_1 and f_2 stand for the continuity and momentum equations. According to the Galerkin method the weighting functions Q , W are set equal to the approximation functions Φ , ψ for the wanted variables h and (u, v) . It shall be noted that the approximation functions Φ , ψ are different for the depth h and the flow velocities (u, v) . The program RMA2 for example uses a linear function for Φ and a square function for ψ .

Partial differentiation and the Gauss theorem are applied to the weighted intergrals from Eq. 3-61. This does **not** apply to the continuity equation since it does not have any second-order derivatives that could be reduced. The derivatives with respect to time are not subject to partial differentiation either (s. chapter 3.1.7). When going through the calculation it is important to watch that the boundary integrals that are formed make sense physically.

For this reason we look at the boundary conditions for Eq. 3-60 again in closer detail, which can be defined in the following way:

$$\begin{aligned} u_i &= \widehat{u}_i(s, t) \quad , \text{with } \widehat{u}_i(s, t) \perp \text{ on } \Gamma_u \\ \mathbf{T}_i &\equiv \boldsymbol{\sigma}_{ij}(s, t) n_j(s) \quad , \text{with } n_j(s) \text{ components of the normal vektors on } \Gamma_T \end{aligned} \quad \text{Eq. 3-62}$$

The set union of the two boundaries Γ_u and Γ_T makes up the whole element boundary Γ_e . T_i are the components of the total shear force at the element boundary, which is composed of the inner friction forces and the (... hydrostatic) pressure. The total shear force T_i or σ_{ij} at the element boundary can be expressed as:

$$\begin{aligned} \sigma_{ij} = \tau_{ij} - p\delta_{ij} = \varepsilon_{ij} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \left(\frac{1}{2} \rho g h^2 \right) \delta_{ij} \quad , \text{where } \delta_{ij} = 1 \quad (i = j) \\ \delta_{ij} = 0 \quad (i \neq j) \end{aligned} \quad \text{Eq. 3-63}$$

With these boundary conditions in mind we approach the weak form of our basis equations, the continuity equation being a simple first practice:

$$0 = \int_{\Omega^e} Q \left(\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} \right) dx dy \quad \text{Eq. 3-64}$$

Afterwards the momentum equation is done at the example of the x-component:

$$\begin{aligned} 0 = \int_{\Omega^e} W \left[h \frac{\partial u}{\partial t} + hu \frac{\partial u}{\partial x} + hv \frac{\partial u}{\partial y} \right] dx dy + \int_{\Omega^e} W \left[g h \frac{\partial z_o}{\partial x} + \frac{\tau_{so,x}}{\rho} \right] dx dy \\ + \underbrace{\int_{\Omega^e} W \left[-\frac{h}{\rho} \varepsilon_{xx} \frac{\partial^2 u}{\partial x^2} - \frac{h}{\rho} \varepsilon_{xy} \frac{\partial^2 u}{\partial y^2} + g \frac{\partial h^2 / 2}{\partial x} \right] dx dy}_{\text{third Integral} \rightarrow \text{further converting}} \end{aligned} \quad \text{Eq. 3-65}$$

Eq. 3-65 is already sorted qualitatively, so only the components of the third integral have to be transformed further. As it can be seen, the second-order derivatives are collected in the third integral, just as the components that are necessary for a physically sensible boundary integral definition. The components of the third integral are processed individually for better understanding, only using the rules of partial differentiation and the gradient theorem.

Third integral, first component:

$$\int_{\Omega^e} W \left[\frac{h}{\rho} \varepsilon_{xx} \frac{\partial^2 u}{\partial x^2} \right] dx dy = \frac{1}{\rho} \left(\int_{\Omega^e} W \left[\frac{\partial}{\partial x} \left(h \varepsilon_{xx} \frac{\partial u}{\partial x} \right) \right] dx dy - \int_{\Omega^e} W \left[\frac{\partial u}{\partial x} \frac{\partial}{\partial x} (h \varepsilon_{xx}) \right] dx dy \right) \text{ Eq. 3-66}$$

The first term of the right side of the equation can be further transformed to:

$$\frac{1}{\rho} \int_{\Omega^e} W \frac{\partial}{\partial x} \left(h \varepsilon_{xx} \frac{\partial u}{\partial x} \right) dx dy = \frac{1}{\rho} \left(\int_{\Omega^e} \underbrace{\frac{\partial}{\partial x} \left(W h \varepsilon_{xx} \frac{\partial u}{\partial x} \right)}_{grad F} dx dy - \int_{\Omega^e} \frac{\partial W}{\partial x} \left(h \varepsilon_{xx} \frac{\partial u}{\partial x} \right) dx dy \right)$$

with theorem of gradient :

Eq. 3-67

$$\int_{\Omega} grad F d\Omega = \oint_{\Gamma} F \vec{n} ds = \oint_{\Gamma} (F \vec{n}_x + F \vec{n}_y) ds$$

$$= \frac{1}{\rho} \left(\oint_{\Gamma^e} \left(W \varepsilon_{xx} h \frac{\partial u}{\partial x} \right) n_x ds - \int_{\Omega^e} \frac{\partial W}{\partial x} \left(h \varepsilon_{xx} \frac{\partial u}{\partial x} \right) dx dy \right)$$

The first component can thus be expressed as:

$$\int_{\Omega^e} W \left[\frac{h}{\rho} \varepsilon_{xx} \frac{\partial^2 u}{\partial x^2} \right] dx dy = \frac{1}{\rho} \left(- \int_{\Omega^e} W \left[\frac{\partial u}{\partial x} \frac{\partial}{\partial x} (h \varepsilon_{xx}) \right] dx dy - \int_{\Omega^e} \frac{\partial W}{\partial x} \left(h \varepsilon_{xx} \frac{\partial u}{\partial x} \right) dx dy + \oint_{\Gamma^e} \left(W \varepsilon_{xx} h \frac{\partial u}{\partial x} \right) n_x ds \right) \text{ Eq. 3-68}$$

Third integral, second component:

In analogy to the first component.

Third integral, third component:

$$\int_{\Omega^e} \mathbf{W} \left[g \frac{\partial h^2}{\partial x} \right] dx dy = -g \int_{\Omega^e} \frac{\partial \mathbf{W}}{\partial x} \frac{h^2}{2} dx dy + g \int_{\Omega^e} \frac{\partial}{\partial x} \left(\mathbf{W} \frac{h^2}{2} \right) dx dy$$

Eq. 3-69

$$= -g \int_{\Omega^e} \frac{\partial \mathbf{W}}{\partial x} \frac{h^2}{2} dx dy + g \oint_{\Gamma^e} \left(\mathbf{W} \frac{h^2}{2} \right) n_x ds$$

With the transformations above that concern the third integral, Eq. 3-65 can now be written in extended form as:

$$0 = \int_{\Omega^e} \mathbf{W} \left[h \frac{\partial u}{\partial t} + hu \frac{\partial u}{\partial x} + hv \frac{\partial u}{\partial y} \right] dx dy + \int_{\Omega^e} \mathbf{W} \left[g h \frac{\partial z_o}{\partial x} + \frac{\tau_{so,x}}{\rho} \right] dx dy$$

$$+ \frac{1}{\rho} \left(\int_{\Omega^e} \mathbf{W} \left[\frac{\partial u}{\partial x} \frac{\partial}{\partial x} (h \epsilon_{xx}) \right] dx dy + \int_{\Omega^e} \frac{\partial \mathbf{W}}{\partial x} \left(h \epsilon_{xx} \frac{\partial u}{\partial x} \right) dx dy - \oint_{\Gamma^e} \left(\mathbf{W} \epsilon_{xx} h \frac{\partial u}{\partial x} \right) n_x ds \right)$$

Eq. 3-70

$$+ \frac{1}{\rho} \left(\int_{\Omega^e} \mathbf{W} \left[\frac{\partial u}{\partial y} \frac{\partial}{\partial y} (h \epsilon_{xy}) \right] dx dy + \int_{\Omega^e} \frac{\partial \mathbf{W}}{\partial y} \left(h \epsilon_{xy} \frac{\partial u}{\partial y} \right) dx dy - \oint_{\Gamma^e} \left(\mathbf{W} \epsilon_{xy} h \frac{\partial u}{\partial y} \right) n_y ds \right)$$

$$- g \int_{\Omega^e} \frac{\partial \mathbf{W}}{\partial x} \frac{h^2}{2} dx dy + g \oint_{\Gamma^e} \left(\mathbf{W} \frac{h^2}{2} \right) n_x ds$$

Which completes the weak form of the x-component of the shallow water equation.

The last step in the deduction of the finite element form is substituting the expressions for the approximate solutions into the weak form. The approximate solutions for the depth h and the flow velocities (u, v) for our unsteady, two-dimensional, depth-integrated problem can be written as follows using a semi-discrete method:

$$u_i(x, y, t) = u_{N,i} = \sum_{m=1}^M \psi_m(x, y) u_i^m(t)$$

Eq. 3-71

$$h(x, y, t) = h_N = \sum_{k=1}^K \phi_k(x, y) h^k(t)$$

ψ_m and ϕ_k are the approximation functions, and u_i^m and h^k are the respective function values at the element vertices with the vertex count M or K respectively. The program RMA2 produces a vertex count of $M=8$ and 6 and $K=4$ and 3 , using square approximation function for the velocities and linear approximation functions for the depth. It allows triangles and rectangles as element shapes. If the approximate solutions from Eq. 3-71 are substituted into the weak form (s. Eq. 3-64 and 3-70) and the assumptions of the Galerkin method are made, i.e. weighting function equals approximation function ($W=\psi$ und $Q=\phi$) the following expressions are obtained:

Continuity:

$$0 = \int_{\Omega^e} \phi \left(\frac{\partial h_N}{\partial t} + \frac{\partial h_N u_N}{\partial x} + \frac{\partial h_N v_N}{\partial y} \right) dx dy$$

Eq. 3-72

Shallow water equation, x-component:

$$\begin{aligned}
 0 = & \int_{\Omega^e} \psi \left[h_N \frac{\partial u_N}{\partial t} + h_N u_N \frac{\partial u_N}{\partial x} + h_N v_N \frac{\partial u_N}{\partial y} \right] dx dy + \int_{\Omega^e} \psi \left[g h_N \frac{\partial z_o}{\partial x} + \frac{\tau_{so,x}}{\rho} \right] dx dy \\
 & + \frac{1}{\rho} \left(\int_{\Omega^e} \psi \left[\frac{\partial u_N}{\partial x} \frac{\partial}{\partial x} (h_N \epsilon_{xx}) \right] dx dy + \int_{\Omega^e} \frac{\partial \psi}{\partial x} \left(h_N \epsilon_{xx} \frac{\partial u_N}{\partial x} \right) dx dy - \oint_{\Gamma^e} \left(\psi \epsilon_{xx} h_N \frac{\partial u_N}{\partial x} \right) n_x ds \right) \\
 & + \frac{1}{\rho} \left(\int_{\Omega^e} \psi \left[\frac{\partial u_N}{\partial y} \frac{\partial}{\partial y} (h_N \epsilon_{xy}) \right] dx dy + \int_{\Omega^e} \frac{\partial \psi}{\partial y} \left(h_N \epsilon_{xy} \frac{\partial u_N}{\partial y} \right) dx dy - \oint_{\Gamma^e} \left(\psi \epsilon_{xy} h_N \frac{\partial u_N}{\partial y} \right) n_y ds \right) \\
 & - g \int_{\Omega^e} \frac{\partial \psi}{\partial x} \frac{h_N^2}{2} dx dy + g \oint_{\Gamma^e} \left(\psi \frac{h_N^2}{2} \right) n_x ds
 \end{aligned}
 \tag{Eq. 3-73}$$

Learning goals in chapter 3 :

Derivation and explanation of FEM:

- What is FEM?
- Which other numerical methods do exist?
- What means discretization in space?
- What means discretization in time? What approaches do exist?
- What is a weighing function? What significance does it have?
- What is an approximation function? What is the difference to the weighing function?
- How is the operation called and which both functions are identical (but are not equivalent)?
- What is a initial condition? Example/Sketch.
- What boundary conditions do you know. Which combinations are possible at flowing waters. Boundary conditions? Example/Sketch.

- Why is one going back to the weak formulation when creating a FE-formulation for the shallow water equation?
- What is Euler forwards and backwards respectively?
- What do the coefficients in the coefficient matrix describe?
- In a water flows a small inflow. Where and how does this inflow occur in the matrix notation respectively in the system of equations?
- What conditions do not apply for model boundaries that are not flowed through and closed?
- Why can the river tube not be discretized in the width with the usage of a FE-element?
- Why can a FE-element not be chosen to be desirably small?
- Which unknowns are substituted by an approximation function in the averaged-depth shallow water equation . Which form must this function have?
- How can **the averaged-depth shallow water equation** be simplified for stagnant waters?

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