3 NUMERICAL METHODS

3.1 FINITE ELEMENT METHOD (FEM)

3.1.1 INTRODUCTION

The Finite Element Method is a numerical method for solving differential equations and integrals, and it is primarily used for problem solving in applied engineering and science. The Finite Element Method is a generalization of the well-established variation approach, which is based on the idea that the solution u of a differential equation can be approximated by a linear combination of the parameter c_j and appropriate functions Φ_j as shown in Eq. 3-1 *[Reddy et al., 1994]*.

$$u \approx u_N \equiv \sum_{j=1}^{N} c_j \Phi_j(x) + \Phi_0 \qquad \text{Eq. 3-1}$$

In Eq. 3-1, u stands for the exact and u_N for the approximated (with FEM) solution. Thus, u stands for the vector of the unknown, which is in this case the water level and current velocities.
The parameters c_j are generally determined with the help of a weighted integral, so that they are a solution of the differential equation of the problem. When selection the functions Φ_j, also called approximation or interpolation functions, it is important that they meet the boundary conditions. There are different methods for the variation approach like for example the Rayleigh-Ritz method or the method of weighted residuals, while the latter can further be distinguished into the Galerkin method, the least square method, and so on. The mentioned methods mainly differ in the choice of the weighting function ψ and the approximation function Φ. The **Galerkin method** for example, which is used in chapter 3.1.8 in the finite element form of the depth-averaged shallow water equation, requires the weighting function to be equal to the approximation function ($\psi = \Phi$). This is the reason why the introduction emphasizes this method.
More detailed explanations about the principles of the variation approach and the corresponding methods can be found in *[Bathe, 1996]* and *[Reddy, 1993]*.

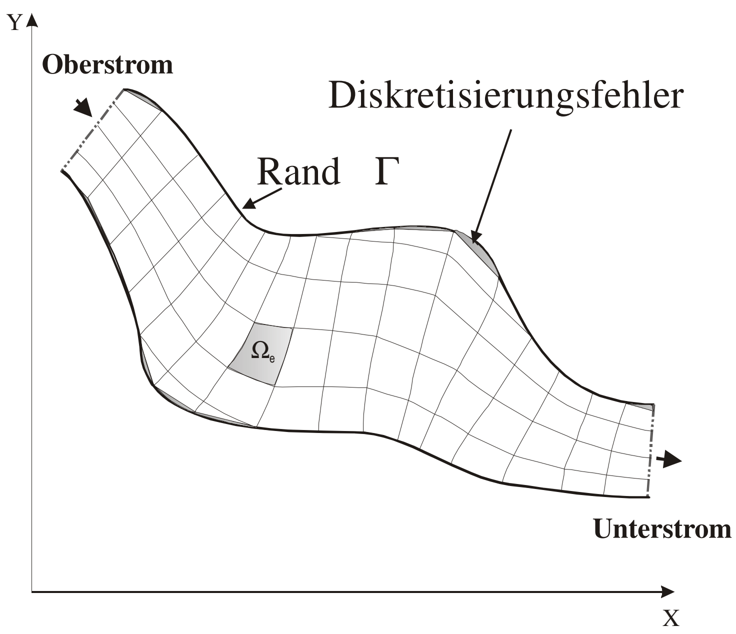

Figure 3-1: Finite element discretisation at the example of a river floodplain.

One of the main disadvantages of the variation approach becomes clear when applying it to complex geometric problems – and most problems in practice are of this type. It is difficult to find an appropriate approximation functions here, because these functions depend on the area geometry, among other dependencies, which will be shown in chapter 3.1.4. Now the finite element principle comes into play. The solution area is divided into a finite number of small sub areas (finite elements), as can be seen in Fig. 3-1. This is also called a finite element discretization of the model area. The geometry of the sub areas is selected to be simple (triangular or rectangular), so that the approximation functions that are necessary for the variation approach can be generated systematically and after a certain pattern. The solving of the differential equation is then related to vertices of the finite elements; if square approximation functions are used, additional vertices are generated on the midpoints of edges and maybe in the centre of elements.

In the following a short introduction to the finite element method shall be given. It does not claim to be a complete and exhaustive explanation of the method however. A number of excellent standard works like for example *[Bathe, 1996]* and *[Reddy, 1993]* exist that can be suggested for further reading. The introduction is rather meant to be a repetition of known material that is supposed to emphasize the main characteristics of the finite element method and that hopefully has an “oh-that-is-how-it-was”-effect for our advanced students. The access to the finite element form of the depth-averaged shallow water equation will probably be easier with this jump start.

The problem analysis prior to the finite element method is divided into the following steps according to *[Reddy et al., 1994]:*

Theory of the Finite Element Method	**Application in practice**
(1) Discretization of the model area into a finite number of finite elements.	Development of a FE-net with river channel and floodplain under consideration of the topographie and the classification of roughness
(2) Derivation of the weighted residual integral or the weak form of the differential equation to investigate and implement the finite element approach for the respective equations.	*(internal program processes)*
(3) Combining the elements in a global, algebraic system of equations.	*(internal program processes)*
(4a) Applying boundary and initial conditions to the system of equations	Involvment and approach of all hydrodynamic boundary conditions (water level-, runoff-hydrograph, W-Q-relations). Choice of a suitable mathematical start solution (lake solution or completely dry)
(4b) Merging area-specific parameters (roughness, eddy viscosity/ turbulence viscosity, fall dry, re-wetting)	Election and levying of all necessary hydraulic parameters. Adjustment within the calibration and/or sensitivity analysis
(5) Solving of the system of equations.	*(internal program processes)* *Monitoring of intermediate results*
(6) Verification, visualisation and validation of the solution	Monitoring, evaluation, visualisation (!) and review of the solution. Visualisation of complex results.

Tabelle 3-1: Work process of a problem analysis using the Finite Element Method

The monitoring and evaluation of erroneous or as well correct results is only possible with the understanding and the knowledge of the “internal program” processes. Only than, if applicable, possible sources of error and wrong assumptions in the modelling can be found.

The points (1)-(3) will be described in chapters 3.1.2 to 3.1.5 by hand of a simple example, while the points (4) and (5) will only be mentioned at the side and in relation to the program RMA2 (respectively RMA10s).

3.1.2 METHOD OF WEIGHTED RESIDUALS → FINITE ELEMENT FORM

This section does not refer to the FEM in general, but only to the specific method of weighted residuals with application of the Galerkin method.

First we have a look at the general differential equation Eq. 3-2, for which we want to find the solution *u*.

$$D(u) = q \qquad \text{Eq. 3-2}$$

D is a linear operator here, in this case a differential operator, and *q* is some kind of outer load. If we substitute the approximation u_N from Eq. 3-1 into Eq. 3-2, the initial equation is not exactly satisfied anymore, and a remainder, also called residual, is generated.

$$R \equiv D(u_N) - q = D\left(\sum_{j=1}^{N} c_j \Phi_j + \Phi_0\right) - q = D\left(\sum_{j=1}^{N} c_j \Phi_j\right) + D(\Phi_0) - q \neq 0 \qquad \text{Eq. 3-3}$$

Assuming *u* to be a function of only *x* and *y* (i.e. a two-dimensional, steady problem), the residual *R* is also a function of *x* and *y*, but also of c_j. With help of the method of weighted residuals, the parameters c_j are chosen so that the residual *R* approaches zero. The weighted integral below has to be solved.

$$\int_{\Omega} \psi_i(x, y) \cdot R(x, y, c_j)\, dx\, dy = 0 \qquad (i = 1, 2, \ldots\ldots, N) \qquad \text{Eq. 3-4}$$

Integration is over the area Ω (two-dimensional area) and ψ_I are the weighting functions, that are principally different from the approximation functions Φ_j. Only for the Galerkin method ψ_i and Φ_j are set equal.

Eq. 3-1 to 3-4 are strictly speaking not a finite element formulation. Eq. 3-1 has to be modified first:

$$u(x, y) \approx u^e(x, y) = \sum_{j=1}^{n} u_j^e\, \psi_j^e(x, y) \qquad \text{Eq. 3-5}$$

where $u^e(x,y)$ is an approximation of the solution ; $u(x,y)$ over the element Ω_e with the vertex count *n;* u_j^e is the value of the function of $u^e(x,y)$ at the vertex *j* of the element and $\psi_j^e(x,y)$ is the approximation function for the element. Note that the definition $\psi = \Phi$ has already been made in Eq. 3-5 according to the Galerkin method.
If Eq. 3-5 is substituted into 3-4, we obtain the following general expression for the finite element form according to the method of weighted residuals:

$$\int_{\Omega^e} \psi_i^e(x, y) \cdot \left\{ D\left(\sum_{j=1}^{n} u_j^e\, \psi_j^e(x, y)\right) + D(\psi_0^e) - q \right\} dx\, dy = 0 \qquad \text{Eq. 3-6}$$